GENERALIZED SEQUENCE SPACES
and
MATRIX TRANSFORMATIONS

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To

My Mother and Father

with

profound affection and respects

MATH-1980-D-PAT

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CERTIFICATE

I am to certify that the research work embodied in the dissertation "Generalized Sequence Spaces and Matrix Transformations" by Mr. John Patterson, M.Sc., M.A., a Ph.D. scholar of this Department, has been carried out under my supervision and that it has not been submitted elsewhere for any degree or diploma.

M.Gupta
(Manjul Gupta)

Date : October 15*, 1980*

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October - 1980 John Patterson

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SYNOPSIS

The theory of vector-valued sequence spaces (VVSS) has emerged widely as a natural generalization of the theory of scalar-valued sequence spaces (SVSS) founded earlier by Köthe and Toeplitz in 1934. An inspiring treatment of this aspect (SVSS) can be seen in the monograph: ''Topological Vector Spaces I; Springer-Verlag (1969)'' by Köthe himself; whereas an elaborate account of this theory including the recent advances as well as those which are useful in constructing examples and counter examples in the duality theory of locally convex spaces, summability domains and the theory of Schauder bases, has been thoroughly unfolded in a recent monograph of Kamthan and Gupta on ''Sequence Spaces and Series; Marcel Dekker, Inc. (1980)''.

Although the vector-valued sequence spaces have their origin in the work of L. Schwartz [Ann. Inst. Fourier 7(1957) 1-141], Grothendieck [Mem. Amer. Math. Soc. 16(1955)], Gel'fand [Mat. Sbornik 4(1938) 235-286] and Phillips [Trans. Amer. Math. Soc. 48(1940) 516-541]; a systematic treatment of the same was given by Pietsch in 1962 in his book ''Verallgemeinerte Vollkommene Folgenraüme; Akademie-Verlag (1962)''. Fascinated by various applications of the theory of SVSS and the importance of VVSS in nuclearity, the study of VVSS was further developed by Gregory, Kimpe, Rosier, Phuong Các, Gupta, Kamthan and Rao during a span of ten years from 1966 to 1976.

Our attempt in this dissertation is to generalize those aspects of SVSS which were possibly untouched so far, e.g., the study of matrix transformations including characterizations of precompact and nuclear diagonal operators on VVSS, duals of generalized sequence spaces and a study of simple generalized sequence spaces. We have also dealt with the generalized forms of ℓ^p-spaces and compact subsets in a VVSS.

The present thesis is divided into six chapters.

Chapter I contains a brief history and development of the theory of SVSS, VVSS and applications of VVSS to nuclear spaces.

Chapter II includes results without proofs from the theory of locally convex spaces, vector-valued sequence spaces and nuclear spaces which are to be used in the subsequent chapters.

Chapter III is entirely devoted to the space $\ell^p(X)$ of absolutely p-summing sequences from a locally convex space X, $1 \leq p < \infty$. We investigate the generalized Köthe and topological duals of these spaces in the most general setting of locally convex spaces and characterize convergence as well as weakly sequentially compact sets therein. We also study the S. Radon-Riesz property in these spaces.

Chapter IV essentially deals with the various concepts of duals of a VVSS. Indeed, we introduce the notions of generalized monotone and symmetric sequence spaces and investigate relationships among α-, β-, γ- and δ- duals of a VVSS $\Lambda(X)$. We also

investigate conditions on $\Lambda(X)$ so that its topological and sequential duals behave like the spaces of generalized sequences. After having introduced the notion of a μ-dual of a VVSS $\Lambda(X)$ corresponding to an SVSS μ, we pass on to a brief discussion on the $\sigma\mu$-topology on a VVSS $\Lambda(X)$. We also deal with the M-character of the dual system $\langle \Lambda(X), \Lambda^{\times}(Y) \rangle$.

Chapter V is mainly concerned with the characterizations of compact sets in a VVSS $\Lambda(X)$.

Chapter VI incorporates a new study concerning the representation of linear operators on VVSS in terms of infinite matrices of operators on the underlying spaces. In other words, we deal with the matrix transformations on VVSS and characterize nuclear and precompact diagonal operators. We also introduce the notion of a generalized simple sequence space and discuss a few topological properties of the same. Finally, we investigate the behaviour of matrix transformations on VVSS which are simple and nuclear.

Chapter 1

HISTORY AND MOTIVATION

1. SCALAR-VALUED SEQUENCE SPACES

The theory of vector-valued sequence spaces (VVSS) has widely emerged as a natural generalization of the theory of scalar-valued sequence spaces (SVSS) founded earlier by Köthe and Toeplitz [71] in 1934 and subsequently developed by Köthe himself in a series of papers [62] through [69]. Thanks to Toeplitz whose suggestion finally inspired Köthe to bring out his monumental contribution in the form of a treatise [70] (cf. [57] also) on topological vector spaces of which the last two sections have stimulated a tremendous amount of interest to initiate researches in the theory of sequence spaces. In this direction, the early contributions due to R.G. Cooke [11], [12] deserve a special mention; infact these books are the first ones containing elementary treatment on the convergence notion in some familiar sequence spaces and transformations thereon related to infinite matrices. The theory of SVSS has provided sufficient motivation to introduce and study several new concepts in the theory of Schauder bases ([26], [59], [75], [112]), structural study of locally convex spaces ([57], [70]), nuclear operators and nuclear spaces ([95], [116]) and summability domains ([7], [123]). Indeed, the SVSS theory has been largely responsible

for constructing examples and counterexamples which the vector space pathologists and Schauder basis experts have found of much use and significance in the study of duality theory of locally convex spaces and the theory of Schauder bases. An heuristic treatment of this theory dealing with these various aspects is contained in a recent monograph [57] of Kamthan and Gupta.

Due to its vast applications, the fascination for the Köthe-Toeplitz theory grew leaps and bounds, thus resulting in its extension and generalization by several analysts. In an attempt to mention briefly the salient features of this theory, let us recall a few standard notations from the theory of SVSS. Throughout we write ω for the class of all sequences $x = \{x_n\}$ with $x_n \in \mathbb{K}$, $n \geq 1$ where $\mathbb{K}$ is the field of scalars; also we use the symbol ϕ for the subspace of ω consisting of finitely many non-zero sequences. By a sequence space (or an SVSS) λ we shall henceforth mean a subspace of ω with $\phi \subset \lambda$.

It was possibly Allen who initiated the task of generalizing several results of Köthe and Toeplitz in 1943. In his paper [2], he introduced the idea of projective convergence of sequences in an SVSS λ which is stronger than the coordinatewise convergence and is weaker than the weak convergence corresponding to the α-dual or Köthe dual $\lambda^\times$ of λ, where

$$\lambda^\times = \{\{y_i\} \in \omega : \sum_{i\geq 1} |x_i y_i| < \infty, \forall \{x_i\} \in \lambda\}.$$

According to him, a sequence $\{x^n\} \subset \lambda$, $x^n = \{x_i^n\}$ is ''projective convergent to x'' relative to another sequence space μ, if

$$\lim_n \sum_{i=1}^{\infty} x_i^n u_i = \sum_{i=1}^{\infty} x_i u_i, \quad \text{for every sequence } \{u_i\} \in \mu .$$

He proved several results on projective and coordinatewise convergence of sequences.

The study of the dual system $\langle \lambda, \lambda^{\times} \rangle$ for a perfect space (that is, $\lambda = \lambda^{\times\times}$) as developed by Köthe and Toeplitz [71] was further carried over to the dual system $\langle \lambda, \lambda^{\beta} \rangle$ by Chillingworth [10], where λ^{β}, known as the β-dual of λ, is defined by

$$\lambda^{\beta} = \{y \in \omega : \sum_{i \geq 1} x_i y_i \text{ converges in } \mathbb{K}, \forall\, x \in \lambda\} .$$

Indeed, the notion of the β-dual of a sequence space λ, which is once again due to Köthe and Toeplitz [71], was exploited by Chillingworth who called it a g-dual and obtained a number of results on matrix transformations on sequence spaces; whereas Mathews used it to extend the theory of matrix rings.

Around the year 1967, Garling (cf. [29] and [30]) developed a general theory of sequence spaces. He introduced another notion of a dual of an SVSS λ; indeed, he defined the γ-dual λ^{γ} of λ as

$$\lambda^{\gamma} = \{y \in \omega : \sup_n \left| \sum_{i=1}^{n} x_i y_i \right| < \infty, \ \forall x \in \lambda\} .$$

He also considered the problem of representing the topological

dual of an SVSS as a sequence space and succeeded in getting such a result through an extensive use of the unit vectors e^i, where $e^i = \{0,0,\ldots,\underset{\substack{\downarrow \\ \text{ith place}}}{1},0,0,\ldots\}$ (We must recall Ruckle's work [106] in this direction as well). Also, in another paper [28], he made a systematic study of symmetric sequence spaces (i.e. the spaces which remain invariant under the permutations of their members) which he [31] carried over further to study the ideals of operators in Hilbert spaces.

To appreciate the scope and limit of symmetric sequence spaces, one comes across with the concept of a δ-dual of an SVSS λ, defined as

$$\lambda^{\delta} = \{x \in \omega : \sum_{i \geq 1} |x_i y_{\rho(i)}| < \infty, \forall\, y \in \lambda \text{ and } \rho \in \pi\},$$

where π is the class of all permutations of natural numbers. This notion was introduced independently by Garling [28] and Ruckle [105], around the years 1966 and 1967. Ruckle [105] also studied the symmetric duals of a single sequence $\{x_i\}$ in ω. It would be appropriate to mention here that Ruckle's earlier study on sequence spaces has mainly been motivated by the theory of Schauder bases. However, in 1972, having observed the roles of various known sequence spaces in defining α-, β-, and γ-duals, he [106] gave a unified idea of a μ-dual of an SVSS λ corresponding to another SVSS μ. According to him,

$$\lambda^{\mu} = \{y \in \omega : \{x_i y_i\} \in \mu , \forall \{x_i\} \in \lambda\} .$$

Using this concept, he generalized many results of Köthe.

The development of the duality theory of sequence spaces had a deep impact on the representation of linear maps on SVSS. Indeed, Köthe and Toeplitz [71] proved that corresponding to the Köthe duality of sequence spaces, every continuous linear transformation from one sequence space λ into another sequence space μ can be given by an infinite matrix $[a_{ij}]$ of scalars. They called a linear operator $A: \lambda \to \mu$, a matrix transformation if A can be given by a matrix $[a_{ij}]$ such that for each $x \in \lambda$, $Ax = y$, where $y_i = \sum_{j\geq 1} a_{ij}x_j$, $i\geq 1$ and $y \in \mu$. In the same paper, they also attempted to study the behaviour of several collections of infinite matrices and introduced the notions of various types of g-rings which according to them, is a collection of infinite matrices closed under matrix addition and multiplication. This study was further carried over by Vermes [118], Copping [14], Weber [120], Mathews [77] etc. However, Allen paid more attention to both these aspects of infinite matrices and made a significant contribution in [3] and [4].

Matrix transformations on several known sequence spaces have been a subject of central investigation by many a mathematician in the past; notably we may mention the work of

Rao [97], Brown [9], Raphael [99] and Hahn [49]. An inherent feature in most of their work is the notion of the ''simple'' character of a sequence space. Indeed, this character of a sequence space was discovered by Jacob [54] in 1977, who exploited it further to get some more results enveloping those obtained by earlier researchers. The paper by Gupta and Kamthan [45] is also worth mentioning in this direction, where they have investigated matrix transformations on an arbitrary simple nuclear sequence space $(\lambda, \eta(\lambda,\lambda^{\times}))$.

A careful study of matrix transformations on sequence spaces reveals that they are certain types of inclusion maps acting from one sequence space into another (cf. [57]). Indeed, if λ, μ are two SVSS and $A = [a_{ij}]$ an infinite matrix, then A transforms λ to μ, means that $\lambda \subset \mu_A$, where μ_A is known as the summability domain of A and is defined as

$$\mu_A = \{x: x \in \lambda \text{ and } Ax \in \mu\} .$$

It has further been observed that the underlying sequence spaces enjoy a universal property referred to as the K-property. These spaces have their origin in the theory of summability. In the process of investigation, various types of sequence spaces, for instance FK-(cf. [123]), conull and co-regular [113], wedge spaces [7] etc. were introduced

and the summability theory became more general and satisfactory by replacing classical matrix arguments with topological methods. Considering specific sequence space in place of μ, especially the space c of convergent sequence, the theory was earlier developed by Agnew [1], Hill [51], Wilanski [121] etc. However, we attribute to Zeller [123], the origin of the general structural study of summability domains. This was further carried over by Bennett [5] and Kalton [55] who tackled the problem related to the representation, BK-, Montel-, reflexive and nuclear characters etc. of a summability domain. As far as the results on inclusion maps are concerned, we may single out the contributions of Zeller [123], Bennett [6], Bennett and Kalton [8], Snyder and Wilanski [114], Kamthan and Gupta [58] and Ruckle [107]. A detailed account of these aspects is to be found in the monograph [57].

The extension of the Köthe-Toeplitz theory to function spaces has been carried out by Cooper [13], Zeller [123], Persson [87], Dieudonné [24], Macdonald ([79], [80], [81]) etc.; whereas Monna [84] and DeGrande-DeKimpe [18] have made their contributions by extending some of the results on perfect sequence spaces to non-Archimidian sequence spaces.

2. VECTOR-VALUED SEQUENCE SPACES

A generalized sequence space or a vector-valued sequence space (VVSS) $\Lambda(X)$ is a vector space containing sequences of vectors

from an arbitrary vector space X as their members. These spaces have their origin in the work of Gel'fand [33] and Phillips [89] who used some abstractly valued function spaces in the characterization of certain linear transformations. Later, in 1951, Grothendieck [39] considered VVSS in the form of tensor products of SVSS ℓ^p with a Banach space X for developing the theory of nuclear spaces. Indeed, the contributions of L.Schwartz [111] and Grothendieck [40] who respectively extended theories of distributions and of holomorphic functions from the field of scalars to an arbitrary locally convex space, have also been a source of motivation to carry out researches in VVSS. In 1960, Roumieu [104] made a brief study of certain type of VVSS in connection with his investigations on generalized distributions.

A major attempt in the direction of generalizing the Köthe theory of perfect sequence spaces to that of VVSS was made by Pietsch in his book [93] in 1962. He combined a perfect SVSS with a locally convex space X in order to define a VVSS $\lambda(X)$ as follows

$$\lambda(X) = \{\{x_i\};\ x_i \in X,\ i \geq 1 \text{ and } \{f(x_i)\} \in \lambda\ ,\ \forall\, f \in X^*\}\ ,$$

where X^* denotes the topological dual of X. After having topologized the space $\lambda(X)$ with the help of the locally convex topology of X and λ, he gave characterizations for various aspects like boundedness, compactness of sets etc. in this space.

Besides, he dealt with the representation of linear operators through abstract sequences, gave examples of some special generalized sequence spaces and pointed out a strong deviation of the theory of VVSS from that of SVSS while studying the dual spaces of such generalized sequence spaces.

Having observed that all the VVSS are not of the type $\lambda(X)$ as introduced by Pietsch, Phoung Các [90] initiated investigations in VVSS which appeared more closely to the study of Köthe and Toeplitz for SVSS referred to above. Indeed, for a dual pair $\langle X,Y\rangle$ of vector-spaces, he considered a vector-space $\Lambda(X)$ of sequences from X with respect to usual coordinate-wise operations and defined another VVSS $\Lambda^{\times}(Y)$, known as the generalized Köthe dual of $\Lambda(X)$ or the associate of $\Lambda(X)$ as

$$\Lambda^{\times}(Y) = \{\{y_i\} : y_i \in Y,\ i \geq 1 \text{ and } \sum_{i\geq 1} |\langle x_i, y_i\rangle| < \infty,\ \forall \{x_i\} \in \Lambda(X)\}.$$

By imposing several restrictions on the dual pair $\langle X,Y\rangle$, he obtained many results for the dual system $\langle \Lambda(X), \Lambda^{\times}(Y)\rangle$ whose counterparts for SVSS are to be found in [57] and [70]. For instance, in connection with the normal hull of weakly compact sets, he proved

''If X is a Fréchet-Montel space, in particular if X is a space of finite dimension, then the normal hull of a weakly compact subset M of a normal space $\Lambda(X)$ containing $\Phi(X)$,

$\Phi(X)$ being the space of all finitely non-zero sequences from X, is weakly compact''.

In his another paper [91], he considered particular type of VVSS which are defined with the help of a countable family of positive sequences in a manner similar to the Köthe spaces of finite and infinite type (cf. [94], p. 97). Having topologized these spaces in a natural way, he proved that their topological duals can be identified with their generalized Köthe duals. He utilized his knowledge of VVSS in his later work [92] in which he could express certain spaces of functions and their duals as the generalized sequence spaces and their generalized Köthe duals.

Almost at the same time, Gregory [38] was working for his doctoral thesis on VVSS. He proved many results for the dual pair $\langle\Lambda(X), \mu(Y)\rangle$, where $\mu(Y)$ is a subspace of $\Lambda^{\times}(Y)$ and $\Phi(Y) \subset \mu(Y)$, on the lines of Phoung Các but in a slightly more general setting. For this pair, he introduced the notion of solid topology (cf. §3, Chapter 2) which includes the normal topology as a particular case and proved under certain assumptions that the Mackey and strong topologies are solid. He also considered those VVSS and the solid topologies whose scalar counterparts were studied earlier by Garling [28]. Besides, he generalized the Köthe-Toeplitz result on matrix transformations and the Grothendieck-Pietsch criterion (cf. [95],p. 98 and [57]) to the setting of VVSS. Concerning the structural

properties of VVSS, he investigated conditions for which a VVSS is complete, quasi-complete, reflexive, semi-reflexive and Montel with respect to the topology arising from the dual pair $\langle \Lambda(X), \mu(Y) \rangle$. For an SVSS λ and a locally convex space X, he introduced a VVSS $\lambda[X]$ defined by

$$\lambda[X] = \{\{x_i\} : x_i \in X,\ i \geq 1 \text{ and } \{p(x_i)\} \in \lambda \text{ , for each continuous semi-norm } p \text{ on } X\},$$

and proved a few properties related to its β-dual.

Around the year 1970, De Grande-De Kimpe observed that the space $\lambda(X)$ of Pietsch doesn't include many interesting examples of generalized sequence spaces, for instance, the space of all absolutely summable sequences in a locally convex space X. She, therefore, introduced the following VVSS [17]

$$\lambda\{X\} = \{\{x_i\} : x_i \in X,\ i \geq 1 \text{ and } \{\sup_{f \in M} |\langle x_i, f \rangle|\} \in \lambda \text{ , for all } M \in \mathfrak{M}\},$$

where $\mathfrak{M}$ is the family of all equicontinuous subsets of the dual X^* of a Hausdorff locally convex space X and λ is a perfect SVSS equipped with a solid topology. Making use of the topologies of λ and X, she defined a locally convex topology on $\lambda\{X\}$ and proved that the topological dual of $\lambda\{X\}$ always contains sequences from the strong dual of X and it is the VVSS $\lambda^{\times}(X^*)$ for a normed space X. By representing $\lambda\{X\}$ as a topological tensor product she proved

'' $\lambda\{X\}$ is nuclear if and only if λ and X are nuclear''. Taking help of the space $\lambda\{X\}$ she considered the mapping of λ-type which are closely related to the nuclear operators, Hilbert-Schmidt operators and absolutely p-summing operators defined by Pietsch [95]. She made use of mappings of λ-type in her later work [19] wherein she treated the ''approximation-property'' of Grothendieck for generalized sequence spaces. In her recent work [23] she applied VVSS to operator ideals and also to obtain information on the structure of a locally convex space X with λ-base, by factoring the identity operator on X through the VVSS $\lambda\{X\}$.

Almost simultaneously, Rosier in his dissertation [103] made a comparative study of the space $\lambda(X)$ of Pietsch and the space $\lambda\{X\}$ which he defined in a manner similar to Gregory [38] and Kimpe [17]. Indeed, corresponding to a certain collection $\mathfrak{M}$ of bounded subsets of $\lambda^{\times}$, the Köthe dual of λ, he considered the spaces $\lambda(X)$ and $\lambda\{X\}$ which are the spaces $\lambda(X)$ and $\lambda\{X\}$ equipped with the topologies (called the $\mathfrak{M}$-topology) generated respectively by the semi-norms:

$$\epsilon_{M,u}(\{x_i\}) = \sup_{\substack{\{\alpha_i\}\in M \\ f\in u^{\circ}}} \sum_{i\geq 1} |\alpha_i|\,|\langle f, x_i\rangle|$$

and

$$\Pi_{M,u}(\{x_i\}) = \sup_{\{\alpha_i\}\in M} \sum_{i\geq 1} |\alpha_i| p_u(x_i),$$

where u is a member of a fundamental system $\mathcal{U}(X)$ of neighbourhoods of $0 \in X$. Analogous to the subspace $[\lambda(X)_{\mathfrak{m}}]$ of $\lambda(X)$, he defined $[\lambda\{X\}_{\mathfrak{m}}]$ as the subspace of $\lambda\{X\}$ containing sequences which are $\mathfrak{m}$-limits of their sections. He discovered that these spaces $[\lambda(X)_{\mathfrak{m}}]$ and $[\lambda\{X\}_{\mathfrak{m}}]$ are very similar relative to their internal construction, e.g., boundedness, compactness, completeness etc.; however there is a marked difference in the duality theory of these spaces. Whereas it is difficult to represent the dual of $[\lambda(X)_{\mathfrak{m}}]$, he proved the following interesting result concerning the dual of $[\lambda\{X\}_{\mathfrak{m}}]$

''The dual of $[\lambda\{X\}_{\mathfrak{m}}]$ is the space of all vectors $\bar{a} = \{a_i\}$, $a_i \in X^*$ which have representations of the form $\bar{a} = \bar{\alpha}\,\bar{u} = \{\alpha_i u_i\}$ with $\bar{\alpha} \in \lambda^\times$ and $\bar{u} = \{u_i\}$, an equicontinuous subset of X^*.''

Finally he related the isomorphisms between the space $[\lambda(X)_{\mathfrak{m}}]$ and $[\lambda\{X\}_{\mathfrak{m}}]$ with the nuclearity of the space λ or X.

It would be appropriate to mention here that the space $\lambda\{X\}$ has independently been studied by Gregory [38], Kimpe [17] and Rosier [103]; however, one finds some overlappings between the results of Kimpe and Rosier.

Almost three years later, Gupta, Kamthan and Rao initiated the task of relating VSSS with the theory of Schauder decomposition in a locally convex space (a Schauder decomposition in an l.c. TVS X is a sequence $\{M_i\}$ of subspaces such that each $x \in X$ is uniquely expressed in the form $x = \sum_{i \geq 1} x_i$, $x_i \in M_i$, $i \geq 1$ and the projections $P_i: X \to X$, $P_i(x) = x_i$, are continuous). They were largely successful in this direction and proved various results relating this aspect of VVSS and the duality roles displayed by many known VVSS in [46], [47], [48]. For instance, concerning Schauder decomposition in a VVSS, they proved (cf. Rao [98] also)

''For a locally convex space X with dual X^*, the VVSS $\Lambda(X)$ always possesses a Schauder decomposition with respect to the weak topology $\sigma(\Lambda(X), \Lambda^{\times}(X^*))$''.

Further, they considered duality relationships between the VVSS $\Phi(X)$, $\Omega(X)$, $\ell^1(X)$, $\ell^\infty(X)$, $c(X)$, $c_0(X)$ etc. and characterized weakly bounded sets in some of these spaces. Besides, they mentioned several examples illustrating various types of Schauder decompositions of VVSS and characterized similar decompositions introduced by them with the help of the associated vector-valued sequence spaces. They related the types of Schauder decompositions with the structure of VVSS in the following form:

''Let X be an l.c. TVS having a Schauder decomposition $\{M_n, P_n\}$ and X^* be its topological dual. Then

(i) $\{M_n\}$ is boundedly complete ==> $\Lambda(X)$ is perfect

(ii) $\{M_n\}$ is bounded multiplier ==> $\Lambda(X)$ is normal

where $\Lambda(X) = \{\{x_n\}: x_n \in M_n$ and $\sum_{i\geq 1} x_i$ converges in $X\}$ ''.

While attempting to solve the question posed by Diestel concerning the Radon-Nikodym property of $\ell^p(X)$, Leonard studied various aspects of the Banach sequence spaces $\ell^p(X)$ in [72]. The contributions of Grabinov [36] [37], Garnir [32] and Patricia Barr [85] also find their due place in the advancement of the theory of VVSS.

A recent Springer-Verlag lecture notes [74] by Maddox reveals that some of the results of SVSS in the direction of summability domains have also been generalized to the setting of VVSS. Indeed, such a study was initiated by Robinson [101] in the year 1950 and was further carried over by Maddox [73], Thorpe [117] and others.

Thus we find that though enough has been done, yet much is left to be accomplished as far as the natural generalization of many results from SVSS to VVSS is concerned. Indeed, the task of generalizing results of SVSS to VVSS has not met with complete success, and we may expect some obvious reasons for it, for instance, one may not define the natural ordering structure in such spaces.

In this dissertation we have tried to bridge several gaps which exist between the SVSS and VVSS. Indeed, we have concentrated on the problem of representing arbitrary linear maps in terms of matrices of linear operators, the representation of dual spaces of VVSS as generalized sequence spaces and various related topics.

3. NUCLEARITY AND VECTOR VALUED SEQUENCE SPACES

As envisaged in the foregoing paragraphs, the theory of vector-valued sequence spaces not only generalizes the results of scalar-valued sequence spaces; but also provides several applications in the theory of Schauder decompositions and the theory of absolutely summing operators and nuclear spaces (cf. [95] for instance). In all the results related to absolutely summing operators and nuclear spaces, the space ℓ^1 of all absolutely summing scalar sequences plays a very prominent role and thus one may expect to have some sort of relationship of nuclearity of the space with the different types of vector-valued ℓ^1-spaces. Before we pass on to present glimpses of the chronological development of the relationship of nuclearity with vector-valued sequence spaces, we would like to sketch a very brief history of nuclear spaces and nuclear operators.

The concept of nuclear spaces heavily depends on the notion of nuclear operators which have their origin in the

work of Schatten [109] and Schatten and von Neumann [110] who studied the same on separable Hilbert spaces. In fact, they called these operators as ''finite trace-class operators'' (in analogy with the matrix transformations from one sequence space into another). Realizing the importance of finite trace-class operators, Grothendieck [41] carried over the study of these operators from Hilbert spaces to Banach spaces and termed them nuclear operators. In the process of studying nuclear operators on Banach spaces and nuclear spaces, Grothendieck used the cumbersome technique of tensor products of locally convex spaces, which was subsequently simplified by Pietsch by utilizing a few vector-valued sequence spaces and their properties. Another approach to study nuclear spaces was given by Mitiagin [83] who made use of the notions of nth diameters of Kolmogorov and diametrical dimension of locally convex spaces due to Kolmogorov [60] and Pelczynski [86].

The recent development in the theory of SVSS especially its duality theory has been largely responsible to pave way for further generalizations of nuclearity of operators and spaces. In fact, much has been done by replacing the space ℓ^1 which has a significant role in the concept of nuclearity, by a number of known sequence spaces. A successful attempt in this direction was first made by Martineau [76] who introduced the idea of s-nuclear operators, s being the space of all rapidly decreasing sequences. This study was further

carried over by a number of mathematicians; and in particular, we may single out the work of [52], [88], [96], [102], [115] and [116] (cf. [56] also for historical development). A unified theory of λ-nuclear operators and spaces is to be found in the memoir [25] of Dubinski and Ramanujan. Recent monograph [57] of Kamthan and Gupta incorporates the study of nuclear sequence spaces via an approach to diametrical dimensions of locally convex spaces.

To be in tune with our main aim, let us now recall the following spaces corresponding to an l.c. TVS X and a perfect sequence space λ, namely,

$$\lambda(X) = \{\{x_i\} \subset X: \{f(x_i)\} \in \lambda , \forall f \in X^*\}$$

and

$$\lambda\{X\} = \{\{x_i\} \subset X: \{p(x_i)\} \in \lambda , \forall p \in D\}$$

where X^* and D respectively stand for the topological dual of X and the family of all continuous semi-norms generating the topology of X.

We begin with a result of Pietsch [95], which characterizes nuclearity of an l.c. TVS in the form of

''An l.c. TVS X is nuclear if and only if $\ell^1(X) \cong \ell^1\{X\}$ ''.

He also proved in [94]

''An l.c. TVS X is nuclear if and only if for a (each) $p \geq 1$, $\ell^p(X) \cong \ell^p\{X\}$ ''.

For a Banach space X, the topological isomorphism of the spaces $\ell^1(X)$ and $\ell^1\{X\}$, leads to the famous theorem of Dvoretzky-Rogers ([95], p. 67), the validity of which by replacing ℓ^1 by any perfect non-nuclear sequence space $(\lambda, \eta(\lambda, \lambda^\times))$ was shown by Rosier; whereas De Grande-De Kimpe sought its partial solution when ℓ^1 is replaced by a suitable class of perfect Banach sequence space (cf. [57] for details).

The relationship of the nuclearity of a space with its base and VVSS is evident in the following result of Kimpe [22].

'Let a Mackey space X have a Schauder base $\{x_n; f_n\}$ such that $\{f_n\}$ is an absolute base for $(X^*, \beta(X^*,X))$ and X^* $\sigma(X^*,X)$-sequentially complete. Then $(X^*,\beta(X^*,X))$ is nuclear $\Longleftrightarrow$ X has property (B) and $\ell^1(X) = \ell^1\{X\} \Longleftrightarrow \ell^1\langle X\rangle = \ell^1(X) \Longleftrightarrow \ell^1\langle X\rangle = \ell^1\{X\}$ and $\{x_i\}$ is a strong basis for X, where the property (B) of an l.c. TVS X means that for every bounded set D of $\ell^1\{X\}$, there exists a bounded set B of X with $\sum_{i\geq 1} ||x_i||_B \leq 1,\ \forall\ \{x_i\} \in D$, $||\ ||_B$ being the gauge of B and $\ell^1\langle X\rangle$ is the space of all Mackey summable sequences in X.'

In her another paper [19] she exploits the nuclearity of λ and X to prove the generalized sequence space character of the quotient space of a VVSS. Precisely we have

''If X and λ are Fréchet nuclear spaces and Y is a closed subspace of X, then $\lambda(X)/\lambda(Y) \cong \lambda(X/Y)$''.

As the wheel of time rolls by, we look forward to the solutions of several outstanding problems relating VVSS with several other aspects of functional analysis. No doubt, this task which will enrich the theory of VVSS and will make it more comprehensive and interesting, seems to be quite challenging.

Chapter 2

PRELIMINARIES

INTRODUCTION

The material of this chapter is necessarily a prelude to our work in subsequent chapters. We incorporate here some of the results from the theory of locally convex spaces, vector-valued sequence spaces and nuclear spaces. All the results are stated without proof and are to be found either in one of the standard texts or monographs, research papers and theses; for instance one may refer to [53], [70], [95], [100], [108] and [122]. The statement of a result which is either from a thesis or a research paper, is preceded or followed by the corresponding reference.

1. LOCALLY CONVEX SPACES

Throughout this work, we use the notation X for a vector space over the field $\mathbb{K}$ of real or complex numbers. We begin with

DEFINITION 1.1: A vector space X equipped with a topology T is called a <u>topological linear space</u> or <u>topological vector space</u> (TVS) if the vector operations are jointly continuous. The topology T is then called a <u>linear</u> or <u>vector topology</u>.

For a linear topology on a vector space X, there exists a fundamental neighbourhood system $\mathcal{U}(X)$ at the origin consisting

of subsets of X, which are absorbing, balanced; and for each $u \in \mathcal{U}(X)$ there corresponds a $v \in \mathcal{U}(X)$ with $v+v \subset u$. In addition, if each $u \in \mathcal{U}(X)$ is also convex, a TVS(X,T) is called a locally convex space (l.c. TVS) and the topology T is said to be a locally convex topology.

Concerning locally convex topology on X, we have

THEOREM 1.2: A topology T on X is a locally convex topology if and only if there exists a family D of T-continuous semi-norms generating T. Precisely the family D of semi-norms is given by $\{p_u : u \in \mathcal{U}(X)\}$ where $p_u(x) = \inf\ \{\alpha > 0 : x \in \alpha u\}$, is the Minkowski functional corresponding to an absorbing, balanced and convex member u of $\mathcal{U}(X)$.

Hereafter we use the symbol D_T, D_X or D (in case there is no confusion likely to arise) for the family of semi-norms generating the topology T of a locally convex space (X,T), $\mathcal{U}(X)$ for the filter base consisting of absorbing, balanced and convex neighbourhoods at origin and $\mathcal{B}$ for the class of all bounded subsets of a locally convex space.

The following is an interesting characterization of bounded sets proved in [38].

LEMMA 1.3: A subset B in an l.c. TVS X is bounded if and only if for every sequence $\{x_i\} \subset B$ and $\{\alpha_i\}$ in ℓ^1, the sequence $(\sum_{i=1}^{n} \alpha_i x_i)_n$ is a Cauchy sequence in X.

A characterization of continuous and equicontinuous family of linear maps in terms of semi-norms is contained in

PROPOSITION 1.4: Let (X,T) and (Y,S) be two l.c. TVS.
(i) A linear map $f:X \to Y$ is continuous if and only if for each $q \in D_S$ there exists $p \in D_T$ and a positive constant M depending on q, such that $q(f(x)) \leq Mp(x)$, for all $x \in X$.
(ii) A family $\mathcal{H}$ of linear maps from X into Y is equicontinuous if and only if for each $q \in D_S$ there exists $p \in D_T$ and $M > 0$ with $q(f(x)) \leq Mp(x)$, for all x in X and f in $\mathcal{H}$.

The well-known Hahn-Banach Theorem is contained in

THEOREM 1.5: Let X be an l.c. TVS, M a subspace of X and f a continuous linear functional defined on M. Then there exists a continuous linear functional g defined on X such that $f(x) = g(x)$ for all $x \in M$.

PROPOSITION 1.6 : For a semi-norm p on a vector space X and $x_o \in X$ there exists a continuous linear functional f on X with $|f(x)| \leq p(x)$ for all $x \in X$ and $f(x_o) = p(x_o)$.

Let us now recall a few results from general topology.

DEFINITION 1.7 : A point x in a topological space X is an adherent point of a net $\{x^\delta : \delta \in \Delta\}$ if for each neighbourhood u of x and $\alpha \in \Delta$, there exists $\beta \in \Delta$ with $\beta \geq \alpha$ and $x^\beta \in u$.

DEFINITION 1.8 : A subset M of a Hausdorff space X is (i) compact if every net in M has at least one adherent point or equivalently every ultrafilter in M converges to a point of M; (ii) countably compact if every sequence in M has an adherent point and (iii) sequentially compact if every sequence in M has a convergent subsequence.

Remark : A subset A of a Hausdorff topological space X is relatively compact if it is contained in a compact set.

In a Hausdorff space X, we have

PROPOSITION 1.9 : (i) Convergent nets have unique adherent points. (ii) If $P:X \to Y$ is an onto map and $\mathcal{F}$ is an ultrafilter in X then $P(\mathcal{F}) = \{P(F):F \in \mathcal{F}\}$ is an ultrafilter in Y. (iii) If a filter $\mathcal{F}$ converges to x, then the corresponding net $\{x^{(F)}: F \in \mathcal{F}\}$ also converges to x; the converse holds for ultrafilters.

PROPOSITION 1.10 : Let f be a continuous map from a Hausdorff space (X,T) to another Hausdorff space (Y,T'). If x is an adherent point of a net $\{x_\alpha: \alpha \in \Delta\}$ in X, then $f(x)$ is an adherent point of the net $\{f(x_\alpha): \alpha \in \Delta\}$ in Y.

PROPOSITION 1.11 : If A is a countably compact subset of a Hausdorff space (X,T) and x is a unique adherent point of a sequence $\{x_n\}$ in A, then $\{x_n\}$ must converge to x relative to the topology T.

2. DUALITY

We denote by $\langle X,Y\rangle$ the <u>dual pair</u> or <u>dual system</u> of two vector spaces X and Y defined over the same field $\mathbb{K}$; the corresponding bilinear functional is denoted by $B(x,y) = \langle x,y\rangle$. The locally convex topology defined on X by the family $\{q_y : y \in Y\}$ of semi-norms, where $q_y(x) = |B(x,y)|$, is called the <u>weak topology</u> which is denoted by $\sigma(X,Y)$. Similarly, we have a weak topology $\sigma(Y,X)$ on Y.

DEFINITION 2.1 : Let $\langle X,Y\rangle$ be a dual pair and $A \subset X$. Then the polar of A denoted by A^o, is the set defined by

$$A^o = \{y \in Y : |B(x,y)| \leq 1 \text{ for all } x \in A\}.$$

For a dual pair $\langle X,Y\rangle$, let $\mathfrak{S}$ be a collection of $\sigma(Y,X)$-bounded subsets of Y. The locally convex topology generated by the family $\{A^o : A \in \mathfrak{S}\}$ or equivalently by the family of seminorms $\{q_A ; A \in \mathfrak{S}\}$, where $q_A(x) \equiv q_{A^o}(x) = \sup_{y \in A} |B(x,y)|$, is called the $\mathfrak{S}$-topology on X. The $\mathfrak{S}$-topology on X coincides with the weak topology $\sigma(X,Y)$ for the collection of finite subsets of Y. If $\mathfrak{S}$ is the collection of <u>all</u> $\sigma(Y,X)$-bounded (resp. all balanced, convex and $\sigma(Y,X)$-compact) subsets of Y, then the corresponding $\mathfrak{S}$-topology on X is called the <u>strong</u> topology (resp. <u>Mackey</u> topology) and is denoted by $\beta(X,Y)$ (res. $\tau(X,Y)$).

Remark : The vector space X equipped with $\tau(X,Y)$ is called a Mackey space.

A locally convex topology T on X is said to be compatible with the dual pair $\langle X,Y\rangle$ if the topological dual of (X,T) is Y. Hereafter, we reserve the symbols X', X^* and X^+ respectively for the algebraic, topological and sequential duals of an l.c. TVS X.

Concerning compatible topologies, we have

PROPOSITION 2.2 : A locally convex topology T on X is compatible with the dual pair $\langle X,Y\rangle$ if and only if $\sigma(X,Y) \leq T \leq \tau(X,Y)$.

PROPOSITION 2.3 : The bounded sets are same for all locally convex topologies compatible with the dual pair $\langle X,Y\rangle$.

PROPOSITION 2.4 : Let X be a locally convex space relative to two compatible Hausdorff topologies T and T' such that $T \geq T'$ and for T there exists a fundamental neighbourhood system at 0, the members of which are complete for T'. Then X is complete for T.

For equicontinuous subsets of X^*, we need

PROPOSITION 2.5 : A subset M of X^* where X is an l.c. TVS, is equicontinuous if and only if $M \subset v^o$ for some $v \in \mathcal{U}(X)$.

PROPOSITION 2.6 : (Alaoglu-Bourbaki): Let X be an l.c. TVS. Then any equicontinuous subset of X^* is $\sigma(X^*,X)$-relatively compact.

Next we define some special kinds of locally convex spaces

DEFINITION 2.7 : Let X be an l.c. TVS. A subset A of X is said to be (i) a barrel if it is absorbing, balanced, convex and closed; and (ii) is bornivorous if it absorbs every bounded set in X.

DEFINITION 2.8 : An l.c. TVS X is said to be (i) barrelled if every barrel in X is a neighbourhood of $0 \in X$; (ii) σ-quasi-barrelled or σ-infrabarrelled, if every $\beta(X^*,X)$-bounded sequence $\{f_i\} \subset X^*$ is equicontinuous; (iii) sequentially barrelled if every sequence $\{f_i\} \subset X^*$ which converges to 0 in $\beta(X^*,X)$, is equicontinuous; and (iv) bornological if every balanced, convex and bornivorous subset of X is a neighbourhood of 0.

Some results concerning barrelled and bornological spaces are stated in

PROPOSITION 2.9 : An l.c. TVS X is barrelled if and only if every $\sigma(X^*,X)$-bounded set in X^* is equicontinuous. Hence for a barrelled space (X,T), $T = \tau(X,X^*) = \beta(X,X^*)$.

PROPOSITION 2.10 : Let X be a barrelled space and $\mathfrak{S}$ be a

collection of bounded subsets of X covering X. Then X^* is quasi-complete for the $\mathfrak{S}$-topology.

THEOREM 2.11 : Let X be a barrelled space and Y be an l.c. TVS. Suppose $\{R_n : n \geq 1\}$ is a sequence of continuous linear maps from X into Y such that $R(x) = \lim_{n\to\infty} R_n(x)$ exists for all $x \in X$. Then R is continuous.

PROPOSITION 2.12 : (i) For a bornological space X, $(X^*, \beta(X^*, X))$ is complete; (ii) every metrizable locally convex space is bornological; and (iii) a complete bornological space is barrelled.

Let (X,T) be a Hausdorff l.c. TVS and X^{**} be the bidual of X, which is defined as the topological dual of X^* corresponding to the topology $\beta(X^*,X)$. For $x \in X$, the linear functional $\tilde{x}$ on X^* given by $\tilde{x}(x^*) = x^*(x)$ for $x^* \in X^*$ is a member of X^{**}. Then the natural cannonical map $J : X \to X^{**}$, $J(x) = \tilde{x}$, is an injection.

Depending on the properties of J, we have

DEFINITION 2.13 : A Hausdorff l.c. TVS is said to be (i) <u>semi-reflexive</u> if J is onto; and (ii) <u>reflexive</u> if J is a topological isomorphism from (X,T) onto $(X^{**}, \beta(X^{**}, X^*))$.

PROPOSITION 2.14 : Every reflexive space is quasi-complete and barrelled.

The following three results on compact sets are due to Eberlein and Šmulian.

PROPOSITION 2.15 : (Šmulian): In a metrizable locally convex space X, weakly sequentially compact and weakly countably compact sets are the same.

PROPOSITION 2.16 : (Eberlein): An (F)-space is reflexive if and only if every bounded subset is weakly relatively countably compact.

PROPOSITION 2.17 : (Eberlein): The weakly closed weakly countably compact subsets of a locally convex space X which is complete for the Mackey topology, are the same as the weakly compact sets.

Recalling the notation R^* for the adjoint of a linear map $R:X \to Y$, we have

PROPOSITION 2.18 : Let (X,T) and (Y,S) be two l.c. TVS and R be a T-S continuous linear map from X into Y. Then R is also $\sigma(X,X^*)$- $\sigma(Y,Y^*)$ continuous.

PROPOSITION 2.19 : Let (X,T_1) and (Y,T_2) be two l.c. TVS. If the linear map $R:X \to Y$ is T_1-T_2 continuous then $R^*: Y^* \to X^*$ is $\sigma(Y^*,Y)$- $\sigma(X^*,X)$ continuous; also if the linear map R is $\sigma(X,X^*)$- $\sigma(Y,Y^*)$ continuous then R is $\tau(X,X^*)$-T_2 continuous. Therefore R is $\sigma(X,X^*)$- $\sigma(Y,Y^*)$ continuous if and only if R is $\tau(X,X^*)$- $\tau(Y,Y^*)$ continuous.

Concerning the sequential dual X^+ of an l.c. TVS (X,T), we have from [119].

DEFINITION 2.20 : A subset B of X^+ is said to be T-limited if $\lim_{n\to\infty} \sup_{f\in B} |f(x_n)| = 0$, for every null sequence $\{x_n\}$ (i.e., $x_n \to 0$) in X. The finest locally convex topology on X having the same convergent sequences as in T, is denoted by T^+. A topological vector space X is called a Mazur space if $X^* = X^+$.

PROPOSITION 2.21 : For a locally convex space (X,T),

$$(X,T^+)^+ = (X,T^+)^* = (X,T)^+.$$

If (X,T) is bornological, then $T = T^+$.

3. VECTOR-VALUED SEQUENCE SPACES

This section provides necessary background from [17], [38], [90] and [93], for our subsequent work.

The vector-valued sequence spaces considered here are defined in two ways. To be precise, for a dual pair $\langle X,Y\rangle$ of vector spaces defined over the same field $\mathbb{K}$, a vector-valued sequence space (VVSS) or a generalized sequence space is a vector space $\Lambda(X)$ of sequences from X with respect to the usual pointwise addition and scalar multiplication. By the generalized Köthe dual $\Lambda^\times(Y)$ and the generalized β-dual $\Lambda^\beta(Y)$ of $\Lambda(X)$, we respectively mean the spaces

$$(\Lambda(X))^{\times} \equiv \Lambda^{\times}(Y) = \{\{y_i\} : y_i \in Y,\ i \geq 1 \text{ and } \sum_{i\geq 1} |\langle x_i, y_i\rangle| < \infty, \text{ for all } \{x_i\} \text{ in } \Lambda(X)\} ;$$

and

$$(\Lambda(X))^{\beta} \equiv \Lambda^{\beta}(Y) = \{\{y_i\} : y_i \in Y,\ i \geq 1 \text{ and } \sum_{i\geq 1} \langle x_i, y_i\rangle \text{ converges for all } \{x_i\} \text{ in } \Lambda(X)\}.$$

The generalized Köthe dual of $\Lambda^{\times}(Y)$ is denoted by $\Lambda^{\times\times}(X) = (\Lambda^{\times}(Y))^{\times}$ and so on.

Corresponding to a vector space X, we define

$$\Omega(X) = \{\bar{x} = \{x_i\} : x_i \in X,\ i \geq 1\} ;$$

$$\Phi(X) = \{\bar{x} = \{x_i\} : x_i \in X,\ i \geq 1 \text{ and } x_i = 0, \text{ for all but finite indices } i\} .$$

We reserve $\bar{x}$, $\bar{y}$ etc. to denote a vector-valued sequence i.e. $\bar{x} = \{x_i\}$ and $\bar{\alpha}$, $\bar{\beta}$ etc. to denote sequences of scalars. The nth section $\bar{x}^{(n)}$ of $\bar{x}$ is defined to be the sequence $\{x_1, x_2, x_3, \ldots, x_n, 0, \ldots\}$. For $x \in X$, the symbol δ_i^x stands for the sequence $\{0, 0, \ldots, x, 0, 0, \ldots\}$ where x is placed at the ith-place. Clearly $\bar{x}^{(n)} = \sum_{i=1}^{n} \delta_i^{x_i}$.

PROPOSITION 3.1 : We always have

(i) $\Phi(Y) \subset \Lambda^{\times}(Y)$

(ii) If $\Delta(X) \subset \Lambda(X)$ then $\Lambda^{\times}(Y) \subset \Delta^{\times}(Y)$

(iii) $\Lambda(X) \subset \Lambda^{\times\times}(X)$

(iv) If $\Phi(X) \subset \Lambda(X)$, then $\Lambda(X)$ and $\Lambda^{\times}(Y)$ form a dual pair $\langle \Lambda(X), \Lambda^{\times}(Y) \rangle$ under the bilinear functional F given by $F(\bar{x},\bar{y}) = \sum_{i \geq 1} \langle x_i, y_i \rangle$, $\bar{x} = \{x_i\} \in \Lambda(X)$ and $\bar{y} = \{y_i\} \in \Lambda^{\times}(Y)$.

The following notions are due to Pietsch [93]

DEFINITION 3.2 : Let M be a subset of $\Lambda(X)$. M is called <u>normal</u> if for $\bar{x} \in M$ and $\{\alpha_i\} \subset \mathbb{K}$, with $|\alpha_i| \leq 1$, $i \geq 1$, the sequence $\bar{\alpha}\,\bar{x} = \{\alpha_i x_i\} \in M$. The <u>normal cover</u> or <u>normal hull</u> of a set M, denoted by $\hat{M}$ is the set of elements $\bar{x}$ such that $\bar{x} = \bar{\alpha}\,\bar{u}$, where $\bar{\alpha} = \{\alpha_i\} \subset \mathbb{K}$, with $|\alpha_i| \leq 1$, $i \geq 1$ and $\bar{u} \in M$. A VVSS $\Lambda(X)$ is said to be <u>perfect</u> if $\Lambda(X) = \Lambda^{\times\times}(X)$. Clearly every perfect VVSS is normal and so $\Lambda^{\times}(Y)$, being perfect is normal.

In order to form the dual pair $\langle \Lambda(X), \Lambda^{\times}(Y) \rangle$, <u>we assume throughout that</u> $\Phi(X) \subset \Lambda(X)$. Thus we have various $\mathfrak{S}$-topologies on either of the spaces corresponding to the dual pair $\langle \Lambda(X), \Lambda^{\times}(Y) \rangle$; for instance the <u>weak</u> topology $\sigma(\Lambda(X),\Lambda^{\times}(Y))$ $(\sigma(\Lambda^{\times}(Y),\Lambda(X)))$, the <u>Mackey</u> topology $\tau(\Lambda(X),\Lambda^{\times}(Y))$ $(\tau(\Lambda^{\times}(Y),\Lambda(X)))$ and the <u>strong</u> topology $\beta(\Lambda(X),\Lambda^{\times}(Y))$ $(\beta(\Lambda^{\times}(Y),\Lambda(X)))$ on $\Lambda(X)$ $(\Lambda^{\times}(Y))$.

There is another natural locally convex topology on $\Lambda(X)$ known as the <u>normal</u> topology which is denoted by $\eta(\Lambda(X),\Lambda^{\times}(Y))$

and is generated by the family $\{p_{\bar{y}} : \bar{y} \in \Lambda^{\times}(Y)\}$ of semi-norms where

$$p_{\bar{y}}(\bar{x}) = \sum_{i\geq 1} |\langle x_i, y_i\rangle|,$$

for each $\bar{y} = \{y_i\}$ in $\Lambda^{\times}(Y)$, $\bar{x} = \{x_i\} \in \Lambda(X)$. Similarly, we define the normal topology $\eta(\Lambda^{\times}(Y), \Lambda(X))$ on $\Lambda^{\times}(Y)$ by the family $\{p_{\bar{x}} : \bar{x} \in \Lambda(X)\}$.

Gregory [38] introduced <u>solid</u> topology on $\Lambda(X)$, which in particular case coincides with the normal topology. He considers an $\mathfrak{S}$-topology on $\Lambda(X)$, generated by the normal hulls of weakly bounded subsets of $\Lambda^{\times}(Y)$. This topology can also be obtained by the family $\{R_S : S \in \mathfrak{S}\}$ of semi-norms, where

$$R_S(\bar{x}) = \sup_{\bar{y}\in S} \sum_{i\geq 1} |\langle x_i, y_i\rangle|, \ \bar{x} \in \Lambda(X).$$

For an l.c. TVS X with dual X^*, Các [90] proves

PROPOSITION 3.3: (i) If a net $\{\bar{x}^{\delta}\}$ in $\Lambda(X)$ converges to $\bar{x}$ in $\sigma(\Lambda(X), \Lambda^{\times}(X^*))$ then $x_i^{\delta} \xrightarrow{\delta} x_i$ in $\sigma(X, X^*)$, for each $i\geq 1$.
(ii) The normal topology $\eta(\Lambda(X), \Lambda^{\times}(X^*))$ on $\Lambda(X)$ is compatible with the dual pair $\langle\Lambda(X), \Lambda^{\times}(X^*)\rangle$.

The following results are from [38] and [43] respectively:

PROPOSITION 3.4 : Let $\langle\Lambda(X), \mu(X^*)\rangle$ be a dual pair of VVSS with $\Lambda(X)$ normal and $\Phi(X^*) \subset \mu(X^*) \subset \Lambda^{\times}(X^*)$. Then $\{\bar{x}^{(n)}\}$ converges to $\bar{x}$, as $n \to \infty$ in $\tau(\Lambda(X), \mu(X^*))$.

PROPOSITION 3.5 : For a normal subspace $\mu(Y)$ of $\Lambda^{\times}(Y)$ such that $\langle\Lambda(X), \mu(Y)\rangle$ forms a dual pair, the $\sigma(\Lambda(X), \mu(Y))$- and $\eta(\Lambda(X), \mu(Y))$-convergent (Cauchy) sequences in $\Lambda(X)$ are the same.

Concerning matrix transformations on VVSS, we have the following due to Gregory [38].

DEFINITION 3.6 : Let $\Lambda(X)$ and $\Lambda(Y)$ be two VVSS corresponding to l.c. TVS X and Y. A linear map $Z:\Lambda(X) \to \Lambda(Y)$ is said to be a <u>matrix transformation</u> from $\Lambda(X)$ to $\Lambda(Y)$ if there is a matrix $[Z_{ij}]$ of linear maps $Z_{ij}: X \to Y$, $i,j \geq 1$ such that for each $\bar{x} \in \Lambda(X)$ the series $\sum_{j\geq 1} Z_{ij}(x_j)$ converges to y_i in Y with respect to $\sigma(Y,Y^*)$ for each $i\geq 1$ and $(Z(\bar{x}))_i = \sum_{j\geq 1} Z_{ij}(x_j) = y_i$.

PROPOSITION 3.7 : Let $(\Lambda^{\times}(X^*), \sigma(\Lambda^{\times}(X^*), \Lambda(X)))$ be sequentially complete and $\mu(Y^*)$ be subspace of $\Lambda^{\times}(Y^*)$ so that $\langle\Lambda(Y), \mu(Y^*)\rangle$ forms a dual pair. Then a linear map $Z:\Lambda(X)\to\Lambda(Y)$ is $\sigma(\Lambda(X),\Lambda^{\times}(X^*))$- $\sigma(\Lambda(Y), \mu(Y^*))$ continuous if and only if it is a matrix transformation of weakly continuous linear maps from X to Y.

For the discussion of another type of VVSS, let us consider a normal SVSS λ and a locally convex space (X,T). Equip X^* ith the topology $\beta(X^*,X)$ and denote by $\mathcal{B}$ the class of all ounded subsets of X. We set

$$\lambda(X) = \{\bar{x} = \{x_i\};\ x_i \in X,\ i \geq 1 \text{ and } \{p(x_i)\} \in \lambda, \text{ for all } p \in D_T\}$$

and

$$\lambda^{\times}(X^*) = \{\bar{f} = \{f_i\};\ f_i \in X^*,\ i \geq 1 \text{ and } \{p_B(f_i)\} \in \lambda^{\times}, \text{ for all } B \in \mathcal{B}\}.$$

Let us assume that λ is equipped with a normal $\mathfrak{S}$-topology T_λ compatible with the dual pair $\langle\lambda, \lambda^{\times}\rangle$ and generated by the semi-norms $\{p_S\colon S \in \mathfrak{S}\}$, where $p_S(\bar{\alpha}) = \sup\limits_{\{\beta_i\} \in S} \{\sum\limits_{i \geq 1} |\alpha_i||\beta_i|\}$ and $\mathfrak{S}$ is a family of normal hulls of balanced, convex $\sigma(\lambda^{\times}, \lambda)$-bounded subsets of $\lambda^{\times}$, covering $\lambda^{\times}$. Then the family $\{p_S \circ p : S \in \mathfrak{S}, p \in D_T\}$ of semi-norms on $\lambda(X)$ given by

$$(p_S \circ p)(\bar{x}) = p_S(\{p(x_i)\}),$$

for $\bar{x} \in \lambda(X)$, generates a Hausdorff locally convex topology on $\lambda(X)$ denoted by $T_{\lambda(X)}$. If λ and X are normed spaces then $\lambda(X)$ is also normed with respect to the norm $\|\ \|_{\lambda(X)}$, $\|\bar{x}\|_{\lambda(X)} = \|\{\|x_i\|_X\}\|_\lambda$. The norm topology on λ is always assumed to be compatible with the dual pair $\langle\lambda, \lambda^{\times}\rangle$ so that $T_\lambda = \tau(\lambda, \lambda^{\times})$.

A semi norm p on λ is said to be <u>absolutely monotone</u> if $p(\bar{\alpha}) \leq p(\bar{\beta})$ whenever $|\alpha_i| \leq |\beta_i|$ for each $i \geq 1$. We shall denote by e^i, $i \geq 1$ the sequence $\{0, 0, \ldots, \underset{\substack{\downarrow \\ \text{ith-place}}}{1}, 0, 0, \ldots\}$

and call a Banach K-space λ as a BK-space.

For the space (λ, T_λ), we would also need the following two results respectively from [38] and [57].

PROPOSITION 3.8 : The topology T_λ on λ is normal if and only if T_λ can be determined by a set of absolutely monotone semi-norms.

PROPOSITION 3.9 : For $\bar{\alpha} \in \lambda$, $\bar{\alpha}^{(n)} \to \bar{\alpha}$ in $\tau(\lambda, \lambda^{\times})$.
Concerning the space $(\lambda(X), T_{\lambda(X)})$ we quote the following from [17].

PROPOSITION 3.10 : (i) $(\lambda(X), T_{\lambda(X)})$ is a GK-GAK- and GC-space. It is a Banach space if X and λ are Banach spaces.

(ii) If X is normed, then $(\lambda(X), T_{\lambda(X)})^* = \lambda^{\times}(X^*)$ where X^* is equipped with the natural norm topology. Further, if λ is also normed then the norms μ and ν defined on $\lambda^{\times}(X^*)$ by

$$\mu(\bar{f}) = \sup_{||\bar{\alpha}||_\lambda \leq 1} \sum_{i \geq 1} ||f_i||_{X^*} |\alpha_i|$$

and

$$\nu(\bar{f}) = \sup_{||\bar{x}||_{\lambda(X)} \leq 1} \left| \sum_{i \geq 1} \langle x_i, f_i \rangle \right|,$$

are equivalent, that is $\lambda^{\times}(X^*)$ is the strong dual of the normed space $\lambda(X)$.

(iii) For the sequence space $(\lambda(X), T_{\lambda(X)})$ the mappings $P_r : \lambda(X) \to \lambda$, defined by $P_r(\bar{x}) = \{r(x_i)\}$, $r \in D$, are uniformly continuous.

4 NUCLEARITY

For the results of this section we refer to [35] and [95]. Let $(X, \| \ \|_X)$ and $(Y, \| \ \|_Y)$ be two normed spaces. Then we have

DEFINITION 4.1 : A linear map $Z: X \to Y$ is said to be

(i) <u>precompact</u> if $Z(B)$ is a precompact subset of Y, where B is the closed unit ball of X; and

(ii) <u>nuclear</u> if there exist sequences $\{y_n\} \subset Y$ and $\{f_n\} \subset X^*$ with $\sum_{n \geq 1} \|y_n\|_Y \|f_n\|_{X^*} < \infty$, such that $Z(x) = \sum_{n \geq 1} \langle x, f_n \rangle y_n$, for all $x \in X$.

For a nuclear map $Z: X \to Y$, its nuclear norm $N(Z)$ is defined by

$$N(Z) = \inf \{ \sum_{n \geq 1} \|y_n\|_Y \|f_n\|_{X^*} \}$$

where the infimum is taken over all possible representations of Z.

Some interesting facts regarding precompact and nuclear maps are stated in

PROPOSITION 4.2 : Let $\{K_n\}$ be a sequence of precompact maps from $(X, \| \ \|_X)$ to $(Y, \| \ \|_Y)$. If $K_n \xrightarrow{n} K$ in the operator norm then K is a precompact map.

PROPOSITION 4.3 : (i) The composition of a nuclear map with a continuous linear map is always nuclear; and

(ii) the pointwise-limit of a Cauchy net in the nuclear norm of nuclear maps is nuclear.

For defining a nuclear space we need construct some normed spaces due to Grothendieck. Indeed, consider $u \in \mathcal{U}(X)$ for a locally convex space X. Then the set u^o is balanced, bounded, convex and $\beta(X^*,X)$-complete subset of X^*. Define $N(u) = \{x \in X, p_u(x) = 0\}$. Then the quotient space $X_u = X/N(u)$, is a normed space with respect to the norm $\hat{p}$ where $\hat{p}(x(u)) = p_u(x)$, $x(u)$ being the equivalence class in X_u corresponding to the element $x \in X$. The subspace $X^*(u^o) = \bigcup_{n=1}^{\infty} n\, u^o$ of X^*, is a Banach space with the norm $p_{u^o}(f) = \sup \{|f(x)| : x \in u\}$. Concerning these spaces, we have

PROPOSITION 4.4 : The Banach space $(X^*(u^o), p_{u^o})$ is the topological dual of $(X_u, \hat{p}_u)$, for $u \in \mathcal{U}(X)$.

Coming back to our discussion on nuclear spaces let $v \in \mathcal{U}(X)$, such that v is absorbed by u. Then there exists a natural continuous embedding $K_u^v : X_v \to X_u$, defined by $K_u^v(x(v)) = x(u)$, where $x(v) \in X_v$ and $x(u) \in X_u$; and hence we have a unique continuous linear extension $\hat{K}_u^v$ of K_u^v, from the completion $\hat{X}_v$ of X_v to the completion $\hat{X}_u$ of X_u.

We are now prepared to give

DEFINITION 4.5 : A locally convex space X is said to be a *nuclear* space if for each $u \in \mathcal{U}(X)$ there exists a $v \in \mathcal{U}(X)$ such that v is absorbed by u and the map $\hat{K}_u^v : \hat{X}_v \to \hat{X}_u$ is nuclear.

The following result due to Gregory [38] is an interesting generalization of the Grothendieck-Pietsch criterion.

PROPOSITION 4.6 : For a locally convex space X, the VVSS $(\Lambda(X), \eta(\Lambda(X), \Lambda^\times(X^*)))$ is nuclear if and only if $\Lambda^\times(X^*) \subset \ell^1 \ \Lambda^\times(X^*)$.

Chapter 3

GENERALIZED ℓ^p-SPACES

1. INTRODUCTION

This chapter is mainly devoted to the study of the space $\ell^p(X)$ $(1 < p < \infty)$ of absolutely p-summing sequences in an l.c. TVS X. Besides a few results on perfect VVSS in section 2, the rest of the sections are related with the study of $\ell^p(X)$ in one way or the other. Indeed, after having investigated the generalized Köthe and topological duals of $\ell^p(X)$ in the most general setting of an l.c. TVS X, we characterize convergence in this space and obtain results concerning the characterization of weakly sequentially compact sets and S-Radon Riesz property.

For an l.c. TVS X, we define

$$\ell^p(X) = \{\bar{x} = \{x_i\}: x_i \in X,\ i \geq 1 \text{ and } \sum_{i\geq 1} (p_u(x_i))^p < \infty, \text{ for each } u \in \mathcal{U}(X)\},\ 1 \leq p < \infty;$$

$$\ell^q(X^*) = \{\bar{f} = \{f_i\}: f_i \in X^*,\ i \geq 1 \text{ and } \sum_{i\geq 1} (p_{B^o}(f_i))^q < \infty, \text{ for each } B \in \mathcal{B}\},\ 1 \leq q < \infty;$$

$$\ell^\infty(X) = \{\bar{x} = \{x_i\}: x_i \in X,\ i \geq 1 \text{ and } \{x_i\} \text{ is bounded in } X\};$$

$$\ell^\infty(X^*) = \{\bar{f} = \{f_i\}: f_i \in X^*,\ i \geq 1 \text{ and } \{f_i\} \text{ is } \beta(X^*,X) \text{ bounded in } X^*\}.$$

2. CHARACTERIZATION OF A PERFECT GENERALIZED SEQUENCE SPACE

The main result of this section, namely the characterization of perfectness of $\Lambda(X)$, has been bifurcated in the form of following two propositions.

PROPOSITION 2.1 : Let X be a weakly complete locally convex space and $\Lambda(X)$ be perfect. Then the space $(\Lambda(X), \eta(\Lambda(X), \Lambda^{\times}(X^*)))$ is complete.

PROOF : Let $\{\bar{x}^{\delta}\}$ be a $\eta(\Lambda(X),\Lambda^{\times}(X^*))$-Cauchy net in $\Lambda(X)$. Then for $\epsilon > 0$ and $\bar{f} = \{f_i\} \in \Lambda^{\times}(X^*)$, we can find $\delta_o \equiv \delta_o(\epsilon,\bar{f})$ such that

$$(*) \qquad \sum_{i\geq 1} |\langle x_i^{\delta}-x_i^{\delta'}, f_i \rangle| < \epsilon, \text{ for } \delta,\delta' \geq \delta_o$$

Now for $f \in X^*$ and $i \in \mathbb{N}$ (the set of all natural numbers), the sequence δ_i^f is in $\Lambda^{\times}(X^*)$. Considering δ_i^f in place of $\bar{f}$ in $(*)$, it follows that the net $\{x_i^{\delta}\}$ is a $\sigma(X,X^*)$-Cauchy net in X, for each $i \geq 1$. As X is $\sigma(X,X^*)$-complete, we have a sequence $\{x_i\} \subset X$ with the property that $x_i^{\delta} \to x_i$ in $\sigma(X,X^*)$ for each $i \geq 1$. For $n \in \mathbb{N}$, we have from $(*)$ that

$$\sum_{i=1}^{n} |\langle x_i^{\delta} - x_i^{\delta'}, f_i\rangle| < \epsilon \quad \text{for } \delta,\delta' \geq \delta_o$$

$$\Longrightarrow \quad \sum_{i=1}^{n} |\langle x_i^{\delta} - x_i, f_i\rangle| < \epsilon, \quad \delta \geq \delta_o$$

on considering the limit over δ'. As n is arbitrary we get

(**) $$\sum_{i\geq 1} |\langle x_i^{\delta} - x_i, f_i\rangle| \leq \epsilon, \ \delta \geq \delta_o$$

Now

$$\sum_{i\geq 1} |\langle x_i, f_i\rangle| \leq \sum_{i\geq 1} |\langle x_i^{\delta_o} - x_i, f_i\rangle| + \sum_{i\geq 1} |\langle x_i^{\delta_o}, f_i\rangle| < \infty$$

$\Longrightarrow$ $\bar{x} \in \Lambda^{\times\times}(X) = \Lambda(X)$, as $\Lambda(X)$ is perfect. Hence (*) and (**) immediately lead to the fact that the net $\{\bar{x}^{\delta}\}$ converges to $\bar{x} = \{x_i\}$ in $\eta(\Lambda(X), \Lambda^{\times}(X^*))$.

Remark : If X is weakly sequentially complete and $\Lambda(X)$ is perfect then $(\Lambda(X), \eta(\Lambda(X), \Lambda^{\times}(X^*)))$ is sequentially complete and hence $(\Lambda(X), \sigma(\Lambda(X), \Lambda^{\times}(X^*)))$ is sequentially complete.

PROPOSITION 2.2 : For an l.c. TVS X, let the VVSS $(\Lambda(X), \eta(\Lambda(X), \Lambda^{\times}(X^*)))$ be sequentially complete. Then $\Lambda(X)$ is perfect.

PROOF : Let $\bar{x} \in \Lambda^{\times\times}(X)$. Then $\sum_{i\geq 1} |\langle x_i, f_i\rangle| < \infty$ for each $\bar{f} \in \Lambda^{\times}(X^*)$. Therefore for $\epsilon > 0$, there exists $n_o \in \mathbb{N}$ such that

$$\sum_{i\geq n_o+1} |\langle x_i, f_i\rangle| < \epsilon$$

and so $\bar{x}^{(n)} \to \bar{x}$ in $\eta(\Lambda^{\times\times}(X), \Lambda^{\times}(X^*))$. As $\bar{x}^{(n)} \in \Lambda(X)$, for each $n \geq 1$ and $\Lambda(X)$ is $\eta(\Lambda(X), \Lambda^{\times}(X^*))$-complete it follows that $\bar{x} \in \Lambda(X)$. Hence $\Lambda^{\times\times}(X) = \Lambda(X)$, since the other inclusion $\Lambda(X) \subset \Lambda^{\times\times}(X)$ is always true.

Combining Propositions 2.1 and 2.2, we get

THEOREM 2.3 : Let X be a weakly complete l.c. TVS. Then $\Lambda(X)$ is perfect if and only if $(\Lambda(X), \eta(\Lambda(X), \Lambda^{\times}(X^*)))$ is complete.

Remark : If X is the space of real or complex numbers in Theorem 2.3, we get the result for scalar-valued sequence spaces (cf. [70] p. 413). In view of this, one can think of the perfectness of VVSS $\lambda(X)$ which are defined with the help of an SVSS λ and an l.c. TVS X, only when λ is perfect. For instance, it is shown in [46] that corresponding to $\lambda = \ell^1$ and ℓ^∞ the spaces $\ell^1(X^*)$ and $\ell^\infty(X)$ are perfect for any l.c. TVS X, whereas $\ell^1(X)$ and $\ell^\infty(X^*)$ are perfect for σ-quasi barrelled spaces. We will show in the next section that the spaces $\ell^p(X)$, $1<p<\infty$, are perfect for sequentially barrelled spaces. However the spaces $\Phi(X)$, $\Phi(X^*)$, $\Omega(X)$ and $\Omega(X^*)$ which are defined independent of any SVSS λ, are always perfect for any l.c. TVS X (see [98], p. 42).

3. DUALS OF $\ell^p(X)$

For a Hausdorff locally convex space X, the spaces $\ell^p(X)$, $1<p<\infty$ and $\ell^q(X^*)$, $1<q<\infty$ are Hausdorff locally convex spaces when they are respectively equipped with the topologies generated by the families $\{P_u : u \in \mathcal{U}(X)\}$ and $\{P_{B^o} : B \in \mathcal{B}\}$ of semi-norms where

$$p_u(\bar{x}) = \left[\sum_{n\geq 1} (p_u(x_n))^p \right]^{1/p}, \quad \bar{x} \in \ell^p(X)$$

and

$$p_{B^o}(\bar{f}) = \left[\sum_{n\geq 1} (p_{B^o}(f_i))^q \right]^{1/q}, \quad \bar{f} \in \ell^q(X^*).$$

One can easily verify that the space $\ell^p(X)$ is metrizable if X is metrizable; $\ell^p(X)$ is complete if X is complete and therefore $\ell^q(X^*)$ would be a complete space for a bornological space X because $(X^*, \beta(X^*, X))$ is complete in this case (cf. Proposition 2.12, Chapter 2). In this section we find the Köthe and topological duals of $\ell^p(X)$ and show that they are nothing but the spaces $\ell^q(X^*)$, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$, for a sequentially barrelled space X. We start with

THEOREM 3.1 : Let X be a sequentially barrelled locally convex space and p,q be such that

$1 < p,q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

(i) $(\ell^p(X))^\times = \ell^q(X^*)$.

(ii) $(\ell^q(X^*))^\times = \ell^p(X)$.

PROOF : (i) For showing $(\ell^p(X))^\times \subset \ell^q(X^*)$, consider $\{f_i\} \in (\ell^p(X))^\times$. Then $\sum_{i\geq 1} |\langle x_i, f_i\rangle| < \infty$, for each $\bar{x} \in \ell^p(X)$. Let $B \in \mathcal{B}$ and $\{\beta_i\}$ be any sequence in ℓ^p. Then we can find a sequence $\{x_i\} \subset B$ such that

$$(*) \qquad \sum_{i\geq 1} |\beta_i|\; p_{B^o}(f_i) \leq \sum_{i\geq 1} |\langle\beta_i x_i, f_i\rangle| + \sum_{i\geq 1} \frac{1}{2^i} .$$

[Indeed, $p_{B^o}(f_i) = \sup_{x\in B} |\langle x,f_i\rangle|$, $\forall\, i \in \mathbb{N}$, and so for $\frac{1}{|\beta_i|2^i}$ (assume $\beta_i \neq 0$, $i \geq 1$) there exists $x_i \in B$ such that

$$p_{B^o}(f_i) \leq |\langle x_i,f_i\rangle| + \frac{1}{|\beta_i|2^i}$$

$$\Longrightarrow \qquad |\beta_i|\; p_{B^o}(f_i) \leq |\langle\beta_i x_i,f_i\rangle| + \frac{1}{2^i} .$$

Since the above inequality is clearly true for the indices i for which $\beta_i = 0$, we conclude (*).] Also for $u \in \mathcal{U}(X)$, there exists $\lambda > 0$ such that $p_u(x) \leq \lambda$, for all $x \in B$ and therefore

$$\begin{aligned} P_u^p(\{\beta_i x_i\}) &= \sum_{i\geq 1} (p_u(\beta_i x_i))^p \\ &= \sum_{i\geq 1} |\beta_i|^p (p_u(x_i))^p \\ &\leq \sum_{i\geq 1} |\beta_i|^p \;\lambda^p < \infty . \end{aligned}$$

Consequently, $\{\beta_i x_i\} \in \ell^p(X)$ and hence from (*), it follows that $\sum_{i\geq 1} |\beta_i|\; p_{B^o}(f_i) < \infty$. As $\{\beta_i\} \in \ell^p$ is arbitrary, we infer $\{p_{B^o}(f_i)\} \in \ell^q$, i.e., $\{f_i\} \in \ell^q(X^*)$.

For proving $\ell^q(X^*) \subset (\ell^p(X))^\times$, let us consider $\{f_i\} \in \ell^q(X^*)$. Then $\sum_{i\geq 1} (p_{B^o}(f_i))^q < \infty$, for every $B \in \mathcal{B}$.

Hence it follows that $f_i \to 0$ as $i \to \infty$, in $\beta(X^*,X)$. As X is sequentially barrelled, $\{f_i\}$ is an equicontinuous subset of X^*. Therefore by Proposition 2.5, Chapter 2, there exists $u \in \mathcal{U}(X)$ such that $\{f_i\} \subset u^o$. Recalling the spaces $(X_u,\hat{p}_u)$ and $(X^*(u^o), p_{u^o})$ of Proposition 4.4, Chapter 2, define $\hat{f}_i \in X^*(u^o)$ by $\hat{f}_i(x(u)) = f_i(x)$, $\forall i \geq 1$. Then $\hat{f}_i$ is continuous linear functional on $(X_u, \hat{p}_u)$. For $\bar{x} \in \ell^p(X)$, consider

$$\begin{aligned}\sum_{i\geq 1} |\langle x_i,f_i\rangle| &= \sum_{i\geq 1} |\langle x_i(u), \hat{f}_i\rangle| \\ &\leq \sum_{i\geq 1} p_{u^o}(f_i)\, \hat{p}_u(x_i(u)) \\ &= \sum_{i\geq 1} p_{u^o}(f_i)\, p_u(x_i).\end{aligned}$$

Since $B \subset \lambda u$ for some $\lambda > 0$, we have $p_{u^o}(f) \leq \lambda\, p_{B^o}(f)$, $\forall f \in X^*(u^o)$. Thus, $\sum_{i\geq 1} |\langle x_i,f_i\rangle| \leq \lambda \sum_{i\geq 1} p_{B^o}(f_i)\, p_u(x_i) < \infty$. Hence $\{f_i\} \in (\ell^p(X))^\times$ and (i) follows

(ii) For proving the inclusion $(\ell^q(X^*))^\times \subset \ell^p(X)$, we will first show that the sequence $\{\alpha_i g_i\} \in \ell^q(X^*)$ for any sequence $\{\alpha_i\}$ in ℓ^q and an equicontinuous sequence $\{g_i\}$ in X^*. Since $\{g_i\}$ is equicontinuous in X^*, there exists $u \in \mathcal{U}(X)$ and a constant $K > 0$ such that

$$|\langle x,g_i\rangle| \leq Kp_u(x), \text{ for all } x \in X,\ i \geq 1.$$

For any $\bar{x} \in \ell^p(X)$, consider

$$\sum_{i\geq 1} |\langle x_i, \alpha_i g_i\rangle| = \sum_{i\geq 1} |\alpha_i| |\langle x_i, g_i\rangle|$$

$$\leq K \sum_{i\geq 1} |\alpha_i| \, p_u(x_i) < \infty$$

Hence $\{\alpha_i g_i\} \in (\ell^p(X))^\times = \ell^q(X^*)$ by part (i).

Let us now consider $\{x_i\} \in (\ell^q(X^*))^\times$ and choose $p_u \in D$. Applying Proposition 1.6, Chapter 2, we get a sequence $\{g_i\} \subset X^*$ such that $|\langle x_i, g_i\rangle| = p_u(x_i)$ and $|\langle x, g_i\rangle| \leq p_u(x)$, $\forall x \in X$ and $i \geq 1$. Thus for $\{\alpha_i\} \in \ell^q$, $\{\alpha_i g_i\} \in \ell^q(X^*)$ and therefore $\sum_{i\geq 1} |\alpha_i| p_u(x_i) = \sum_{i\geq 1} |\alpha_i| |\langle x_i, g_i\rangle| < \infty$. Hence $\{p_u(x_i)\} \in \ell^p$, that is $\{x_i\} \in \ell^p(X)$.

In order to show $\ell^p(X) \subset (\ell^q(X^*))^\times$, consider $\{x_i\} \in \ell^p(X)$. Let $\{f_i\}$ be an arbitrary element of $\ell^q(X^*)$. Then $\{f_i\}$ is an equicontinuous subset of X^* and therefore $\{f_i\} \subset u^o$, for some $u \in \mathcal{U}(X)$. Once again considering the spaces $(X_u, \hat{p}_u)$ and $(X^*(u^o), p_{u^o})$ and proceeding as in part (i), we conclude that $\sum_{i\geq 1} |\langle x_i, f_i\rangle| < \infty$. Thus $\{f_i\} \in (\ell^q(X^*))^\times$ and hence $\ell^p(X) = (\ell^q(X^*))^\times$.

Since the numbers p and q are interchangeable, the above result immediately leads to

COROLLARY 3.2 : For a sequentially barrelled space X and $1<p,q<\infty$, the spaces $\ell^p(X)$ and $\ell^q(X^*)$ are perfect.

Concerning the topological duals, we have

THEOREM 3.3 : If X is sequentially barrelled and $1 < p$, $q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$(\ell^p(X))^* = \ell^q(X^*)$$

where $\ell^p(X)$ is equipped with the locally convex topology generated by the family $\{P_u : u \in \mathcal{U}(X)\}$ of semi-norms.

PROOF: We first show that $(\ell^p(X))^* \subset \ell^q(X^*)$. Let us first observe that the map $R_i : X \to \ell^p(X)$, $R_i(x) = \delta_i^x$ is an algebraic isomorphism for each $i \geq 1$. Since $P_u(R_i(x)) = p_u(x)$, each map R_i, $i \geq 1$, is also continuous. By fixing i, X can be regarded as a closed subspace of $\ell^p(X)$. Now, consider $F \in (\ell^p(X))^*$ and define $\{f_i\} \subset X'$ by $\langle x, f_i \rangle = F(\delta_i^x)$, $i \geq 1$. Since F is continuous, $\{f_i\} \subset X^*$.

For $\bar{x} \in \ell^p(X)$, we know that $\bar{x}^{(n)} \to \bar{x}$ in $\ell^p(X)$. Indeed, for $\epsilon > 0$, we can find $n_o \in \mathbb{N}$ such that $\sum_{i \geq n_o+1} (p_u(x_i))^p < \epsilon$. Hence $P_u^p(\bar{x}^{(n)} - \bar{x}) \to 0$, as $n \to \infty$. Therefore $F(\bar{x}^{(n)}) = \sum_{i=1}^{n} F(\delta_i^{x_i})$, implies

$$(*) \qquad F(\bar{x}) = \sum_{i \geq 1} \langle x_i, f_i \rangle .$$

Define $\{\alpha_i\} \subset \mathbb{K}$ such that $\alpha_i \langle x_i, f_i \rangle = |\langle x_i, f_i \rangle|$. As $\ell^p(X)$ is perfect by Corollary 3.2 and $|\alpha_i| = 1$, $i \geq 1$, $\{\alpha_i x_i\} \in \ell^p(X)$. Thus by $(*)$, $\sum_{i \geq 1} \langle \alpha_i x_i, f_i \rangle$ converges, i.e.,

$\sum_{i\geq 1} |\langle x_i, f_i\rangle| < \infty$. Consequently, $\{f_i\} \in (\ell^p(X))^\times = \ell^q(X^*)$ by Theorem 3.2 and therefore $(\ell^p(X))^* \subset \ell^q(X^*)$.

For reverse inclusion, let us consider $\{f_i\} \in \ell^q(X^*)$. By Theorem 3.2, the series $\sum_{i\geq 1} \langle x_i, f_i\rangle$, being absolutely convergent for every $\bar{x} \in \ell^p(X)$, is convergent and therefore we can define a map F from $\ell^p(X)$ to $\mathbb{K}$ by

$$F(\bar{x}) = \sum_{i\geq 1} \langle x_i, f_i\rangle.$$

Clearly, F is a linear functional on $\ell^p(X)$.

For showing the continuity of F, we observe that the sequence $\{f_i\}$ is equicontinuous in X^* and therefore $\{f_i\} \subset u^o$, for some $u \in \mathcal{U}(X)$. As every bounded set is absorbed by u, it obviously follows that $\{p_{u^o}(f_i)\} \in \ell^q$. Now applying Proposition 4.4, Chapter 2 and proceeding as in the proof of part (i) of Theorem 3.1, we get

$$\begin{aligned}|F(\bar{x})| &\leq \sum_{i\geq 1} |\langle x_i, f_i\rangle| \\ &\leq \sum_{i\geq 1} p_{u^o}(f_i)\, p_u(x_i) \\ &\leq \Big(\sum_{i\geq 1} (p_{u^o}(f_i))^q\Big)^{1/q} \Big(\sum_{i\geq 1} (p_u(x_i))^p\Big)^{1/p}\end{aligned}$$

$$\Longrightarrow \quad |F(\bar{x})| \leq K\, P_u(\bar{x}),$$

where $K = \Big(\sum_{i\geq 1} (p_{u^o}(f_i))^q\Big)^{1/q}$. Since the map F defined

by $\{f_i\}$ is continuous, each $\{f_i\}$ gives rise to a continuous linear functional F on $\ell^p(X)$. Thus we conclude $F \equiv \{f_i\} \in (\ell^p(X))^*$. This completes the proof.

For $\ell^1(X)$, we have

THEOREM 3.4 : If X is a σ-quasi-barrelled space, then $(\ell^1(X))^* = \ell^\infty(X^*)$, where $\ell^1(X)$ is equipped with the topology generated by the family $\{P_u : u \in \quad (X)\}$ of semi-norms.

PROOF : Since X is σ-quasi barrelled, each sequence in $\ell^\infty(X^*)$, being $\beta(X^*,X)$-bounded, is equicontinuous. Now the proof runs on the lines similar to that of Theorem 3.3.

Remark : For $p = \infty$, the result is not true even in the case when $X = \mathbb{K}$. Indeed, we cannot claim the convergence of $\bar{x}^{(n)}$ to $\bar{x}$ in the topology of $\ell^\infty(X)$.

4 CONVERGENCE IN $\ell^p(X)$

This section starts with

THEOREM 4.1 : Let (X,T) be a sequentially barrelled l.c. TVS and $1 < p < \infty$. If $\{\bar{x}^\delta\}$ is a net in $\ell^p(X)$ and $\bar{x} \in \ell^p(X)$, then $\bar{x}^\delta \to \bar{x}$ in the topology of $\ell^p(X)$ if and only if

(i) $x_i^\delta \to x_i$ in X relative to T, $\forall i \geq 1$; and

(ii) $P_u(\bar{x}^\delta) \to P_u(\bar{x})$, for all $u \in \mathcal{U}(X)$.

PROOF : If $\bar{x}^{\delta} \to \bar{x}$ in $\ell^p(X)$, then (i) follows from the inequality

$$p_u(x_i^{\delta} - x_i) \leq \left(\sum_{i\geq 1} (p_u(x_i^{\delta}-x_i))^p\right)^{1/p} = P_u(\bar{x}^{\delta} - \bar{x})$$

which holds for each $p_u \in D_T$. The condition (ii) is immediate from the inequality

$$|P_u(\bar{x}) - P_u(\bar{y})| \leq P_u(\bar{x} - \bar{y}),$$

where $\bar{x},\bar{y}$ are in $\ell^p(X)$.

Let us now assume that (i) and (ii) hold. As $\bar{x} \in \ell^p(X)$, we have $\sum_{i\geq 1} (p_u(x_i))^p < \infty$, $\forall u \in \mathcal{U}(X)$. Therefore, for given $\epsilon > 0$ and $u \in \mathcal{U}(X)$, we can find a positive integer $N_o \equiv N_o(\epsilon,u)$ such that

$$\sum_{i>N_o} (p_u(x_i))^p < \frac{\epsilon^p}{8 \cdot 2^{p-1}} .$$

Let us now consider,

$$\begin{aligned}
[P_u(\bar{x}^{\delta} - \bar{x})]^p &= \sum_{i\geq 1} (p_u(x_i^{\delta} - x_i))^p \\
&= \sum_{i=1}^{N_o} (p_u(x_i^{\delta}-x_i))^p + \sum_{i>N_o} (p_u(x_i^{\delta} - x_i))^p \\
&\leq \sum_{i=1}^{N_o} (p_u(x_i^{\delta}-x_i))^p + \sum_{i>N_o} [p_u(x_i^{\delta}) + p_u(x_i)]^p \\
&\leq \sum_{i=1}^{N_o} (p_u(x_i^{\delta}-x_i))^p + 2^{p-1} \Big[\sum_{i>N_o} (p_u(x_i^{\delta}))^p \\
&\qquad + \sum_{i>N_o} (p_u(x_i))^p \Big]
\end{aligned}$$

the inequality $(a+b)^p \leq \max(2^{p-1},1)(a^p+b^p)$, $a,b\geq 0$ (cf.[95]).

Making use of (i), we can find δ' such that

$$\sum_{i=1}^{N_o} (p_u(x_i^\delta - x_i))^p < \frac{\epsilon^p}{2}, \ \delta \geq \delta'.$$

From (ii), we have

$$\sum_{i>N_o} (p_u(x_i^\delta))^p \xrightarrow{\delta} \sum_{i>N_o} (p_u(x_i))^p.$$

Hence we can find δ'', such that

$$\sum_{i>N_o} (p_u(x_i^\delta))^p - \sum_{i>N_o} (p_u(x_i))^p < \frac{\epsilon^p}{4.2^{p-1}}, \ \delta \geq \delta''.$$

Choose δ_o such that $\delta_o \geq \delta', \delta''$. Then for $\delta \geq \delta_o$,

$$[p_u(\bar{x}^\delta - \bar{x})]^p \leq \frac{\epsilon^p}{2} + 2^{p-1}[\frac{\epsilon^p}{4.2^{p-1}} + 2 \sum_{i>N_o} (p_u(x_i))^p]$$

$$< \frac{\epsilon^p}{2} + 2^{p-1} [\frac{\epsilon^p}{4.2^{p-1}} + 2 \ \frac{\epsilon^p}{8.2^{p-1}}] = \epsilon^p.$$

Thus $\bar{x}^\delta \to \bar{x}$ in the topology of $\ell^p(X)$ and hence the result is true.

Concerning the weak convergence in $\ell^p(X)$, $1 < p < \infty$, we have

THEOREM 4.2 : Let X be a sequentially barrelled l.c. TVS and $1 < p < \infty$. Assume $\bar{x} \in \ell^p(X)$ and $\{\bar{x}^n\}$ is a sequence in $\ell^p(X)$. Then $\bar{x}^n \to \bar{x}$ weakly in $\ell^p(X)$ if and only if

(i) $x_i^n \to x_i$ in $\sigma(X,X^*)$, $\forall i \geq 1$; and

(ii) for every $u \in \mathcal{U}(X)$, there exists an $M_u > 0$ such that $(\sum_{i\geq 1} (p_u(x_i^n))^p)^{1/p} < M_u$, $\forall n \geq 1$.

PROOF : Let $\bar{x}^n \to \bar{x}$ in $\sigma(\ell^p(X), \ell^q(X^*))$. Since $\delta_i^f \in \ell^q(X^*)$ for each $f \in X^*$ and $i \geq 1$, we have from hypothesis that $\langle \bar{x}^n - \bar{x}, \delta_i^f \rangle = \langle x_i^n - x_i, f \rangle \to 0$ as $n \to \infty$; and hence (i) holds. Now the sequence $\{\bar{x}^n\}$, being weakly convergent in $\ell^p(X)$, is $\sigma(\ell^p(X), \ell^q(X^*))$-bounded. Also from Theorem 3.3 the topologies $\sigma(\ell^p(X), \ell^q(X^*))$ and $T_{\ell^p(X)}$ are compatible and so $\{\bar{x}^n\}$ is bounded in $T_{\ell^p(X)}$ by Proposition 2.3, Chapter 2. Therefore for each P_u, there exists $M_u > 0$, satisfying (ii).

Let us now assume that (i) and (ii) hold. It follows from (ii) that for each $u \in \mathcal{U}(X)$ we can find a constant $K_u = M_u + P_u(\bar{x})$ such that

$$P_u(\bar{x}^n - \bar{x}) \leq K_u, \quad \forall n \geq 1$$

and so, the set $\{\bar{x}^n - \bar{x}, n \geq 1\}$ is a bounded set in $\ell^p(X)$.

For each $i \geq 1$, let us denote by M_i, the subspace $\{\delta_i^f : f \in X^*\}$ of $\ell^q(X^*)$. Then $\overline{sp}\{\bigcup_{i\geq 1} M_i\} \subset \ell^q(X^*)$, where the closure is considered in the topology of $\ell^q(X^*)$. In order to show the equality $\ell^q(X^*) = \overline{sp}\{\bigcup_{i\geq 1} M_i\}$, let us take B in $\mathcal{B}$ and an element $\{f_i\} \in \ell^q(X^*)$. Then

$$(*) \qquad P_{B^o}(\{f_i\}) = \Big(\sum_{i\geq 1} (p_{B^o}(f_i))^q\Big)^{1/q} < \infty.$$

Consider

$$P_{B^o}(\bar{f} - \bar{f}^{(n)}) = P_{B^o}(0,0,\ldots,f_{n+1}, f_{n+2},\ldots)$$

$$= \Big(\sum_{i>n} (p_{B^o}(f_i))^q\Big)^{1/q} \to 0 \text{ as } n \to \infty,$$

in $\ell^q(X^*)$, by virtue of $(*)$. Since $\bar{f}^{(n)} = \sum_{i=1}^{n} \delta_i^{f_i} \in sp\{\bigcup_{i\geq 1} M_i\}$, for each $n \geq 1$, it follows that $\{f_i\} \in \overline{sp}\{\bigcup_{i\geq 1} M_i\}$; and hence $\overline{sp}\{\bigcup_{i\geq 1} M_i\} = \ell^q(X^*)$.

Now from (i) $\langle \bar{x}^n - \bar{x}, \delta_i^f\rangle = \langle x_i^n - x_i, f\rangle \to 0$, as $n \to \infty$, for all $f \in X^*$ and $i \geq 1$. Since a member $\bar{h}$ in $sp\{\bigcup_{i\geq 1} M_i\}$ is a finite linear combination of elements in $\bigcup\{M_i : i \geq 1\}$, $\langle \bar{x}^n - \bar{x}, \bar{h}\rangle \to 0$, as $n \to \infty$.

Consider now $\bar{f} = \{f_i\} \in \ell^q(X^*)$. Then for a bounded set B in X and $\epsilon > 0$, we can find $\bar{h} \in sp\{\bigcup_{i\geq 1} M_i\}$ such that $P_{B^o}(\bar{f}-\bar{h}) < \epsilon$. Since $\{f_i\}$ is an equicontinuous set in X^* and $h_i = 0$ for all except finitely many indices i, it follows that the set $\{f_i - h_i, i \geq 1\}$ is contained in u^o, for some $u \in \mathcal{U}(X)$. Also $B \subset \lambda u$ for some $\lambda > 0$, implies that $p_{u^o} \leq \lambda p_{B^o}$. Once again we use the spaces $(X_u, \hat{p}_u)$ and $(X^*(u^o), p_{u^o})$ given in Proposition 4.4, Chapter 2, and consider

$$\Big|\sum_{i\geq 1} \langle x_i^n - x_i, f_i\rangle\Big|$$

$$\leq \sum_{i\geq 1} |\langle x_i^n - x_i, f_i - h_i\rangle| + \sum_{i\geq 1} |\langle x_i^n - x_i, h_i\rangle|$$

$$\leq \sum_{i\geq 1} p_u(x_i^n - x_i)\, p_{u^o}(f_i - h_i) + \sum_{i\geq 1} |\langle x_i^n - x_i, h_i\rangle|$$

$$\leq \lambda \sum_{i\geq 1} p_u(x_i^n - x_i)\, p_{B^o}(f_i - h_i) + \sum_{i\geq 1} |\langle x_i^n - x_i, h_i\rangle|$$

$$< \lambda \Big(\sum_{i\geq 1} (p_u(x_i^n - x_i))^p\Big)^{\frac{1}{p}} \Big(\sum_{i\geq 1} (p_{B^o}(f_i - h_i))^q\Big)^{\frac{1}{q}} + \sum_{i\geq 1} |\langle x_i^n - x_i, h_i\rangle|$$

$$= \lambda\, P_u(\bar{x}^n - \bar{x})\, P_{B^o}(\bar{f} - \bar{h}) + \sum_{i\geq 1} |\langle x_i^n - x_i, h_i\rangle|$$

For the above $\epsilon > 0$, we can find an integer $n_o \in \mathbb{N}$ such that

$$\sum_{i\geq 1} |\langle x_i^n - x_i, h_i\rangle| < \epsilon, \text{ for all } n \geq n_o.$$

Hence

$$\Big|\sum_{i\geq 1} \langle x_i^n - x_i, f_i\rangle\Big| \leq (\lambda K_u + 1)\,\epsilon, \text{ for all } n \geq n_o.$$

As $\{f_i\} \in \ell^q(X^*)$ is arbitrary, the weak convergence of $\bar{x}^n$ to $\bar{x}$, follows.

5. WEAK SEQUENTIAL COMPACTNESS IN $\ell^p(X)$

The characterization of compact sets in the space $\ell^p(X)$ is an immediate consequence of a general result proved in [17], [38] and [103]; however, we characterize weakly relatively sequentially compact sets in Theorem 5.2, in which we make use of

LEMMA 5.1 : In an l.c. TVS (X,T), if a sequence $\{x_n\}$ converges weakly to x, then

$$\liminf_{n\to\infty} p_u(x_n) \geq p_u(x),$$

for every $u \in \mathcal{U}(X)$.

PROOF : Let $\epsilon > 0$ be an arbitrarily chosen small number and $p_u \in D_T$ for $u \in \mathcal{U}(X)$. Since $p_u(x) = \sup\{|f(x)| : f \in u^o\}$, we can find $f_o \in u^o$ such that

$$|f_o(x)| > p_u(x) - \epsilon/2, \forall x \in X.$$

Also, $x_n \to x$, as $n \to \infty$ in $\sigma(X,X^*)$ implies $f_o(x_n) \to f_o(x)$, as $n \to \infty$. Hence for the above $\epsilon > 0$, there exists $n_o \in \mathbb{N}$ such that for all $n \geq n_o$, $||f_o(x)| - |f_o(x_n)|| < \epsilon/2$, which implies that

$$|f_o(x_n)| > |f_o(x)| - \epsilon/2, \forall\, n \geq n_o.$$

Since $f_o \in u^o$, we have $f_o(x_n) \leq p_u(x_n)$, $n \geq 1$ and so

$$p_u(x_n) > |f_o(x)| - \epsilon/2, \forall\, n \geq n_o.$$

This in turn, implies

$$p_u(x_n) > p_u(x) - \epsilon, \forall\, n \geq n_o.$$

Hence

$$\liminf_{n \to \infty} p_u(x_n) \geq p_u(x).$$

The main result of this section is

THEOREM 5.2 : Let X be a sequentially barrelled l.c. TVS and $1 < p < \infty$. Then a subset A of $\ell^p(X)$ is weakly relatively sequentially compact if and only if

(i) A is bounded in $\ell^p(X)$; and

(ii) $C_n(A)$ is weakly relatively sequentially compact set in X, $\forall n \geq 1$, where $C_n : \ell^p(X) \to X$ are linear maps defined by $C_n(\bar{x}) = x_n$, $\bar{x} = \{x_n\} \in \ell^p(X)$.

PROOF : Let A be a weakly relatively sequentially compact set in $\ell^p(X)$. Then A is weakly bounded and therefore bounded in $\ell^p(X)$ (cf. Proposition 2.3, Chapter 2). For proving (ii) we fix an integer $n_o \in \mathbb{N}$ and consider $C_{n_o}(A)$. Let $\{y_n\}$ be a sequence in $C_{n_o}(A)$. Then we can find a sequence $\{\bar{x}^n\}$ in A such that $x_{n_o}^n = y_n$, $\forall n \geq 1$. As A is weakly relatively sequentially compact, we can find a sequence $\{\bar{x}^{n_j}\} \subset \{\bar{x}^n\}$ such that $\bar{x}^{n_j} \to \bar{x}$, as $j \to \infty$, in $\sigma(\ell^p(X), \ell^q(X^*))$. Thus for each $\{f_i\} \in \ell^q(X^*)$,

$$\left|\sum_{i\geq 1} \langle x_i^{n_j} - x_i, f_i\rangle\right| \to 0, \text{ as } j \to \infty.$$

Now for $f \in X^*$ and $i \geq 1$, $\delta_i^f \in \ell^q(X^*)$ and so choosing δ_i^f in place of $\{f_i\}$, in particular, we get

$$|\langle x_i^{n_j} - x_i, f\rangle| \to 0, \text{ as } j \to \infty, \forall i \geq 1.$$

This clearly implies that $y_{n_j} = x_{n_o}^{n_j} \to x_{n_o}$ as $j \to \infty$ weakly in X. Hence $C_{n_o}(A)$ is weakly relatively sequentially compact in X and so is $C_n(A)$, for all $n \geq 1$.

For converse, let us assume that (i) and (ii) are true. Consider a sequence $\{\bar{x}^n\}$ in A. As $\{x_1^n\}$ is contained in $C_1(A)$, which is weakly relatively sequentially compact in X, we can select a subsequence $n_i(1)$ of $\mathbb{N}$ such that $x_1^{n_i(1)}$ converges to x_1 as $i \to \infty$ relative to $\sigma(X, X^*)$. Now $\{x_2^{n_i(1)} : i \geq 1\}$ is contained in $C_2(A)$, implies that we can select a subsequence

$\{x_2^{n_i(2)} : i \geq 1\}$ of $\{x_2^{n_i(1)} : i \geq 1\}$ such that $x_2^{n_i(2)} \to x_2$, as $i \to \infty$ in $\sigma(X,X^*)$ for some $x_2 \in X$. Continuing in this manner, we get a subsequence $\{x_j^{n_i(j)} : i \geq 1\}$ of $\{x_j^{n_i(j-1)} : i \geq 1\}$ such that $x_j^{n_i(j)} \to x_j$, as $n \to \infty$ in $\sigma(X,X^*)$, for some $x_j \in X$, where $j \in \mathbb{N}$. If we write $\bar{x} = \{x_i\}$ then the subsequence $\{\bar{x}^{n_i(i)} : i \geq 1\}$ of $\{\bar{x}^n\}$ is such that $x_j^{n_i(i)} \to x_j$, as $i \to \infty$ weakly in X, for all $j \geq 1$. To prove that $\bar{x} \in \ell^p(X)$, let $p_u \in D$ for $u \in \mathcal{U}(X)$ and choose $N \in \mathbb{N}$ arbitrarily. Since A is bounded in $\ell^p(X)$, there exists $M_u > 0$ such that

$$\sum_{j=1}^{N} (p_u(x_j^{n_i(i)}))^p \leq M_u^p \text{ , for all } i \geq 1.$$

Applying Lemma 5.1, we have

$$(p_u(x_j))^p \leq \liminf_{i \to \infty} (p_u(x^{n_i(i)}))^p$$

$$\Longrightarrow \quad \sum_{j=1}^{N} (p_u(x_j))^p \leq \sum_{j=1}^{N} \liminf_{i \to \infty} (p_u(x^{n_i(i)}))^p$$

$$\leq \liminf_{i \to \infty} \sum_{j=1}^{N} (p_u(x^{n_i(i)}))^p \leq M_u^p$$

As the above inequality is true for all $N \geq 1$, we conclude that $\bar{x} = \{x_i\} \in \ell^p(X)$. Now applying Theorem 4.2, we get the result.

Remark : For a reflexive Fréchet space X, the space $\ell^p(X)$, $1 < p < \infty$ is also reflexive ($\boxed{17}$, p. 142) and Fréchet. Therefore, the applications of the results of Eberlein and Šmulian (cf Propositions 2.15, 2.16 and 2.17, Chapter 2) yield

THEOREM 5.3 : Let X be a reflexive Fréchet space and A be a subset of $\ell^p(X)$, $1 < p < \infty$. Then the following statements are equivalent :

(i) A is weakly compact.

(ii) A is weakly countably compact.

(iii) A is weakly sequentially compact.

(iv) A is bounded and for every sequence $\{\bar{x}^n\}$ in A such that $x_i^n \to x_i$ as $n \to \infty$ in $\sigma(X,X^*)$, for each $i \geq 1$, the sequence $\bar{x} = \{x_i\} \in A$.

PROOF : (i) ==> (ii) is true from definition. For (ii) ==> (iii) apply Proposition 2.15, Chapter 2. Let us prove (iii) ==> (iv) and (iv) ==> (i).

(iii) ==> (iv). Suppose A is weakly sequentially compact and $\{\bar{x}^n\}$ is a sequence in A such that $x_i^n \to x_i$, as $n \to \infty$ in $\sigma(X,X^*)$, for all $i \geq 1$. As A is weakly sequentially compact, we can find a subsequence $\{\bar{x}^{n_j}\}$ of $\{\bar{x}^n\}$ such that $\bar{x}^{n_j} \to \bar{y}$, as $j \to \infty$ in $\sigma(\ell^p(X), \ell^q(X^*))$, for some $\bar{y} \in A$. Hence $x_i^{n_j} \to y_i$, as $j \to \infty$ in $\sigma(X,X^*)$, for all $i \geq 1$. From the hypothesis and the Hausdorff character of $\sigma(X,X^*)$ we get $\bar{x} = \bar{y}$ and hence $\bar{x} \in A$.

(iv) ==> (i). Let us observe that $\ell^p(X)$, being a reflexive F-space, is barrelled and complete relative to the Mackey topology $\tau(X,X^*)$ (cf. Propositions 2.9 and 2.14, Chapter 2). Now, let A be a subset of $\ell^p(X)$ satisfying (iv). Then in view of Propositions 2.16 and 2.17, Chapter 2, it suffices to prove

that A is weakly sequentially closed. Let $\bar{x}$ be a weak sequential limit point of A. Then there exists a sequence $\{\bar{x}^n\}$ in A such that $\bar{x}^n \to \bar{x}$, as $n \to \infty$ in $\sigma(\ell^p(X), \ell^q(X^*))$. This implies that $x_i^n \to x_i$, as $n \to \infty$ in $\sigma(X,X^*)$, for each $i \geq 1$ and hence $\bar{x} \in A$ by (iv). This completes the proof of the theorem.

6. S-RADON RIESZ PROPERTY

We introduce

DEFINITION 6.1 : An l.c. TVS X is said to satisfy the S-Radon Riesz property if for a sequence $\{x_n\}$ in X and $x \in X$ such that $x_n \to x$, as $n \to \infty$ in $\sigma(X,X^*)$ and $p_u(x_n) \to p_u(x)$, as $n \to \infty$, for each $u \in \mathcal{U}(X)$, it follows that $p_u(x_n - x) \to 0$, as $n \to \infty$, that is $x_n \to x$, as $n \to \infty$ relative to the topology of X.

According to the classical Radon-Riesz theorem (cf. [50], p. 233), the spaces ℓ^p, $1 < p < \infty$ possess this property. However for the spaces $\ell^p(X)$, $1 < p < \infty$, such a result is true when X is uniformly convex Banach space (cf. [16] and [82]). In this section, we prove that $\ell^p(X)$, $1 < p < \infty$, where X is a sequentially barrelled space, has this property if and only if X has the same; which is indeed a generalization of the corresponding result of Leonard [72]. We prove

THEOREM 6.2 : Let X be a sequentially barrelled space and $1 < p < \infty$. Then $\ell^p(X)$ has the S-Radon Riesz property if and only if X has the S-Radon-Riesz property.

PROOF : Let us first assume that $\ell^p(X)$ has the S-Radon Riesz property. Consider a sequence $\{x_n\}$ in X and an element x in X such that $x_n \to x$, as $n \to \infty$ in $\sigma(X,X^*)$ and $p_u(x_n) \to p_u(x)$, as $n \to \infty$ for each $u \in \mathcal{U}(X)$. Define

$$\bar{y}^n = \{x_n, 0, 0, \ldots\ldots\} \qquad \forall n \geq 1$$

and

$$\bar{y} = \{x, 0, 0, \ldots\ldots\ldots\} .$$

Then $\bar{y}^n$, $\bar{y}$ are in $\ell^p(X)$ and $P_u(\bar{y}^n) \to P_u(\bar{y})$ as $n \to \infty$, for each $u \in \mathcal{U}(X)$. Also, for $\bar{f} = \{f_i\} \in \ell^q(X^*) = (\ell^p(X))^*$, the equality $\langle \bar{y}^n - \bar{y}, \bar{f} \rangle = \langle x_n - x, f_1 \rangle$ and $\sigma(X,X^*)$-convergence of x_n to x imply that $\bar{y}^n$ converges to $\bar{y}$ weakly in $\ell^p(X)$. Therefore from hypothesis, $P_u(\bar{y}^n - \bar{y}) \to 0$ as $n \to \infty$ and hence $p_u(x_n - x) = P_u(\bar{y}^n - \bar{y}) \to 0$ as $n \to \infty$, for each $u \in \mathcal{U}(X)$; that is, X has the S-Radon Riesz property.

For converse, we assume that X has the S-Radon Riesz property. Let $\bar{y}^n, \bar{y} \in \ell^p(X)$ be such that $\bar{y}^n \to \bar{y}$, as $n \to \infty$ weakly in $\ell^p(X)$ and $P_u(\bar{y}^n) \to P_u(\bar{y})$, as $n \to \infty$, for each $u \in \mathcal{U}(X)$. Then in view of Theorem 4.1 and our hypothesis about the space X, it suffices to prove that $p_u(y_i^n) \to p_u(y_i)$, as $n \to \infty$, for each $u \in \mathcal{U}(X)$ and $i \geq 1$.

Let us, therefore, consider a member u of $\mathcal{U}(X)$. Since the sequence $\{\bar{y}^n\}$, being weakly convergent in $\ell^p(X)$, is bounded in $\ell^p(X)$; we can find a constant M_u, depending on u, such that

$$\Big(\sum_{i\geq 1} (p_u(y_i^n)^p)\Big)^{\frac{1}{p}} < M_u^{\frac{1}{p}},$$

for all $n \geq 1$. This, in particular, implies that

$$(p_u(y_i^n))^p \leq M_u$$

for all $n \geq 1$ and $i \geq 1$. If we write $\bar{b}^n = \{b_i^n\}$, where $b_i^n = (p_u(y_i^n))^p$, then $\{\bar{b}^n\}$ is a sequence in the space $[0,M_u]^{\mathbb{N}}$, which is the countable product space of the closed intervals $[0,M_u]$. As each $[0,M_u]$ is compact and metrizable, the space $[0,M_u]^{\mathbb{N}}$ is also compact and metrizable. Hence there exists an element $\bar{b} = \{b_i\}$ in $[0,M_u]^{\mathbb{N}}$ and a subsequence $\{\bar{b}^{n_j}\}$ of $\{\bar{b}^n\}$ such that $\bar{b}^{n_j} \to \bar{b}$, as $j \to \infty$. As the convergence in the product topology is equivalent to the coordinatewise convergence, it follows that $(p_u(y_i^{n_j}))^p = b_i^{n_j} \to b_i$, as $n_j \to \infty$, for each $i \geq 1$. Since $y_i^{n_j} \to y_i$, as $n_j \to \infty$ weakly in X, applying Lemma 5.1, we get

$$(*) \qquad \lim_{n_j \to \infty} (p_u(y_i^{n_j}))^p \geq (p_u(y_i))^p, \quad \text{for each } i \geq 1.$$

In order to prove $\lim_{n_j \to \infty} (p_u(y_i^{n_j}))^p = (p_u(y_i))^p$, we assume that for some $i_o \in \mathbb{N}$,

$$\lim_{n_j \to \infty} (p_u(y_{i_o}^{n_j}))^p > (p_u(y_{i_o}))^p.$$

Choose $\epsilon > 0$ such that

$$(**) \qquad \lim_{n_j \to \infty} (p_u(y_{i_o}^{n_j}))^p > (p_u(y_{i_o}))^p + \epsilon.$$

Since $\bar{y} \in \ell^p(X)$, there exists a positive integer $N > i_o$ such that

$$\sum_{i>N} (p_u(y_i))^p < \epsilon/2.$$

Therefore,

$$\sum_{i=1}^{N} (p_u(y_i))^p > \sum_{i\geq 1} (p_u(y_i))^p - \epsilon/2.$$

As $N > i_o$, it follows from (*) and (**) that

$$\lim_{n_j \to \infty} \sum_{i=1}^{N} (p_u(y_i^{n_j}))^p > \sum_{i=1}^{N} (p_u(y_i))^p + \epsilon$$

$$> \sum_{i\geq 1} (p_u(y_i))^p + \epsilon/2.$$

Since right hand side is independent of N, we get

$$\lim_{n_j \to \infty} \sum_{i\geq 1} (p_u(y_i^{n_j}))^p \geq \sum_{i\geq 1} (p_u(y_i))^p + \frac{\epsilon}{2},$$

or

$$\lim_{n_j \to \infty} (P_u(\bar{y}^{n_j}))^p \geq (P_u(\bar{y}))^p + \frac{\epsilon}{2}.$$

This contradicts that $P_u(\bar{y}^n) \to P_u(\bar{y})$, as $n \to \infty$. Hence

$$\lim_{n_j \to \infty} (p_u(y_i^{n_j}))^p = (p_u(y_i))^p, \quad \forall\ i \geq 1.$$

Consequently, $b_i = (p_u(y_i))^p$, for all $i \geq 1$. Thus, we observe that if we consider any subsequence of $\{\bar{b}^n\}$, it contains

another subsequence converging to the same point $\bar{b}$. Hence it follows that $b_i^n \to b_i$ as $n \to \infty$, for every $i \geq 1$. Therefore, $p_u(y_i^n) \to p_u(y_i)$ as $n \to \infty$, for every $i \geq 1$ and $u \in \mathcal{U}(X)$. This establishes the result.

Chapter 4

DUALS OF GENERALIZED SEQUENCE SPACES

1. INTRODUCTION

This chapter is devoted to the study of various duals of a VVSS $\Lambda(X)$. Starting our discussion with the introduction of various terms required to prove the results of this chapter, we proceed to relate the generalized α-, β-, γ- duals of $\Lambda(X)$ in section 3. In sections 4 and 5, we investigate conditions to be laid down on a topological VVSS $\Lambda(X)$ so that its topological and sequential duals themselves behave as generalized sequence spaces, for in that case one can conveniently develop a good deal of duality theory between $\Lambda(X)$ and its topological dual. In the final section of this chapter, we discuss a few aspects of relationship of $\Lambda(X)$ with its μ-dual and study in brief the impact of $\sigma\mu$-topology on $\Lambda(X)$, introduced earlier in section 2 corresponding to an SVSS μ. We also introduce the notion of a completely bounded set in $\Lambda(X)$ and prove a result which yields the M-character of a dual system $\langle \Lambda(X), \Lambda^{\times}(Y)\rangle$.

2 TERMINOLOGY

We start this section with a notion defined in [97].

DEFINITION 2.1 : For a VVSS $\Lambda(X)$, where X is a vector space,

and J a subsequence of $\mathbb{N}$ (set of all natural numbers), the space

$$\Lambda_J(X) = \{\bar{x} = \{x_i\}: \text{there is a } \bar{u} = \{u_i\} \text{ in } \Lambda(X) \text{ such that } x_i = u_{n_i} \text{ for all } n_i \in J\}$$

is called the J-<u>step space</u> of $\Lambda(X)$. The <u>cannonical pre-image</u> of an element $\bar{x}_J$ in $\Lambda_J(X)$ is the sequence $\tilde{x}_J$ which agrees with the coordinates of $\bar{x}_J$ on the indices in J and is zero elsewhere. The <u>cannonical pre-image</u> of $\Lambda_J(X)$ is the space $\tilde{\Lambda}_J(X)$ containing cannonical pre-images of all elements of $\Lambda_J(X)$.

For an SVSS λ and a VVSS $\Lambda(X)$, we define

$$\lambda \cdot \Lambda(X) = \{\{\alpha_i x_i\}: \{\alpha_i\} \in \lambda \text{ and } \{x_i\} \in \Lambda(X)\}.$$

We introduce

DEFINITION 2.2 : A VVSS $\Lambda(X)$ is called (i) <u>symmetric</u> if $\bar{x}_\pi = \{x_{\pi(i)}\} \in \Lambda(X)$ whenever $\bar{x} \in \Lambda(X)$ and $\pi \in \mathcal{P}$ (all permutations of $\mathbb{N}$); and (ii) <u>monotone</u> if $\Lambda(X)$ contains the cannonical pre-images of all its step spaces.

For a dual pair $\langle X,Y\rangle$ of vector spaces X and Y, the <u>generalized</u> γ-<u>dual</u> $\Lambda^\gamma(Y)$ of $\Lambda(X)$ is the space defined by

$$\Lambda^\gamma(Y) = \{\{y_i\}: y_i \in Y,\ i \geq 1 \text{ and } \sup_n \left|\sum_{i=1}^{n} \langle x_i, y_i\rangle\right| < \infty, \text{ for all } \{x_i\} \in \Lambda(X)\}.$$

Corresponding to various permutations of the set $\mathbb{N}$, we have another notion of a generalized dual of $\Lambda(X)$, known as the <u>generalized δ-dual</u> of $\Lambda(X)$ and defined as follows

$$\Lambda^{\delta}(Y) = \{\{y_i\}: y_i \in Y,\ i \geq 1 \text{ such that } \sum_{i\geq 1} |\langle x_i, y_{\pi(i)}\rangle| < \infty, \text{ for all } \{x_i\} \in \Lambda(X);\ \pi \in \mathcal{P}\}$$

Note : The following inclusions among generalized α-, β-, γ- and δ-duals of a VVSS $\Lambda(X)$ are easily verified

$$\Lambda^{\delta}(Y) \subset \Lambda^{\times}(Y) \subset \Lambda^{\beta}(Y) \subset \Lambda^{\gamma}(Y)$$

Let μ be an SVSS equipped with a Hausdorff locally convex topology T_μ generated by the family D_μ of all T-continuous semi-norms. Corresponding to the dual pair $\langle X,Y\rangle$, we have

DEFINITION 2.3 : The <u>generalized</u> μ-<u>dual</u> of a VVSS $\Lambda(X)$ is the space $(\Lambda(X))^{\mu}$ defined as follows:

$$(\Lambda(X))^{\mu} \equiv \Lambda^{\mu}(Y) = \{\bar{y} \in \Omega(Y): \{\langle x_i, y_i\rangle\} \in \mu, \text{ for all } \bar{x} \text{ in } \Lambda(X)\}.$$

For $\mu = \ell^1$, cs and bs (cf. [57], [70]), the corresponding μ-duals are α-, β-, and γ-duals respectively.

DEFINITION 2.4 : A VVSS $\Lambda(X)$ is called μ-<u>perfect</u> if $\Lambda(X) = \Lambda^{\mu\mu}(X) \equiv (\Lambda^{\mu}(Y))^{\mu}$.

Note : For $\mu = \ell^1$, the ℓ^1-perfect spaces are perfect spaces defined in Chapter 2. Clearly every μ-dual of a VVSS $\Lambda(X)$ is always μ-perfect.

In general, a VVSS $\Lambda(X)$ corresponding to a dual pair $\langle X,Y\rangle$ is not necessarily in duality with its μ-dual $\Lambda^{\mu}(Y)$. However we can topologize the space $\Lambda(X)$ with the help of the family D_{μ} and the members of $\Lambda^{\mu}(Y)$. Indeed, for $\bar{y} \in \Lambda^{\mu}(Y)$ and $p \in D_{\mu}$, define

$$p_{\bar{y}}(\bar{x}) = p\,(\{\langle x_i,y_i\rangle\}), \text{ for } \bar{x} \in \Lambda(X).$$

The locally convex topology on $\Lambda(X)$ generated by the family $\{p_{\bar{y}}: \bar{y} \in \Lambda^{\mu}(Y),\ p \in D_{\mu}\}$ of semi-norms on $\Lambda(X)$, is <u>denoted</u> by $\sigma\mu(\Lambda(X), \Lambda^{\mu}(Y))$ and if there is no confusion likely to arise, we will abbreviate this notation as $\sigma\mu$. Similarly we can define the $\sigma\mu$-topology on $\Lambda^{\mu}(Y)$ with the help of semi-norms $\{p_{\bar{x}}: p \in D_{\mu},\ \bar{x} \in \Lambda(X)\}$, where $p_{\bar{x}}(\bar{y}) = p(\{\langle x_i,y_i\rangle\})$. Clearly, the $\sigma\mu$- topology on $\Lambda(X)$ coincides with the normal topology $\eta(\Lambda(X),\Lambda^{\times}(Y))$ when $\mu = \ell^1$.

For a locally convex space X and a VVSS $(\Lambda(X),\mathcal{F})$ equipped with a locally convex topology $\mathcal{F}$, we introduce the following

DEFINITION 2.5 : (i) A VVSS $(\Lambda(X),\mathcal{F})$ is said to be a GK-<u>space</u> if the map $P_i: \Lambda(X) \to X$, $P_i(\bar{x}) = x_i$, $\bar{x} \in \Lambda(X)$ is continuous

for each $i \geq 1$. A GK-space is called a (ii) GAD-space if $\Phi(X)$ is dense in $\Lambda(X)$; and (iii) GAK-space if for each $\bar{x} = \{x_i\}$ in $\Lambda(X)$ $\bar{x}^{(n)} \to \bar{x}$ as $n \to \infty$, relative to $\mathcal{F}$ (iv) $(\Lambda(X), \mathcal{F})$ is said to be a GC-space (resp. GSC-space) if the map $R_i : X \to \Lambda(X)$, $R_i(x) = \delta_i^x$, $x \in X$, is continuous (resp. sequentially continuous) for each $i \geq 1$.

Note : When $X = \mathbb{K}$, the foregoing definitions (i) through (iii) correspond to an SVSS λ which is then called a K-space, an AD-space and an AK-space respectively (cf. [28] and [57]).

3. GENERALIZED α-, β-, γ- and δ-DUALS OF $\Lambda(X)$

A useful characterization of a normal and monotone VVSS $\Lambda(X)$ in terms of SVSS ℓ^∞ and m_o defined respectively as the space of all bounded sequences and the space spanned by the set of all sequences formed by zero's and ones, is contained in

PROPOSITION 3.1 : For a vector space X, a VVSS $\Lambda(X)$ is
(i) normal if and only if $\ell^\infty \circ \Lambda(X) \subset \Lambda(X)$; and
(ii) monotone if and only if $m_o \circ \Lambda(X) \subset \Lambda(X)$.

PROOF : (i) Let us first assume that $\ell^\infty \circ \Lambda(X) \subset \Lambda(X)$ and $\{\alpha_i\} \subset \mathbb{K}$ such that $|\alpha_i| \leq 1$, $i \geq 1$. Then $\{\alpha_i\} \in \ell^\infty$ and therefore $\{\alpha_i\}\{x_i\} = \{\alpha_i x_i\} \in \Lambda(X)$ for $\bar{x} \in \Lambda(X)$. Now let $\Lambda(X)$ be normal, $\bar{x} \in \Lambda(X)$ and $\{\alpha_i\} \in \ell^\infty$. Then $|\alpha_i| \leq M$, $i \geq 1$ implies that $|\alpha_i/M| \leq 1$, $i \geq 1$ and so $\{(\alpha_i/M)x_i\} \in \Lambda(X)$. Hence $\{\alpha_i x_i\} \in \Lambda(X)$, and therefore $\ell^\infty \circ \Lambda(X) \subset \Lambda(X)$.

(ii) Let $\Lambda(X)$ be monotone. Consider $\bar{\alpha} = \{\alpha_i\}$ in m_o and $\bar{y} = \{y_i\}$ in $\Lambda(X)$. Let $\beta_1,\beta_2,\ldots,\beta_n$ be the scalars assumed by α_i's repeatedly over the subsets $J_1,J_2,\ldots,J_n$ of $\mathbb{N}$ and $i_1,i_2,\ldots,i_p$ are the finite indices corresponding to distinct α_i's. Then

$$\tilde{y}_{J_i} \in \tilde{\Lambda}_{J_i}(X) \quad \text{for} \quad i=1,2,3,\ldots,n \;;$$

and

$$\bar{\alpha}\,\bar{y} = \{\alpha_i y_i\} = \sum_{i=1}^{n} \beta_i\, \tilde{y}_{J_i} + \sum_{j=1}^{p} \alpha_{i_j}\, \delta_{i_j}^{y_{i_j}} .$$

Hence $\bar{\alpha}\,\bar{y} \in \Lambda(X)$ and therefore $m_o \circ \Lambda(X) \subset \Lambda(X)$.

Conversely, let $m_o \circ \Lambda(X) \subset \Lambda(X)$ and J be any subsequence of $\mathbb{N}$. Then for $\bar{x} \in \tilde{\Lambda}_J(X)$, we can find $\{y_i\} \in \Lambda(X)$ such that $x_i = y_i$, for $i \in J$ and $x_i = 0$ for $i \in \mathbb{N}-J$. Define $\bar{\alpha} = \{\alpha_i\}$ by

$$\alpha_i = \begin{cases} 1 & i \in J \\ 0 & i \in \mathbb{N}-J \end{cases}$$

Then $\bar{\alpha} \in m_o$ and $\bar{x} = \bar{\alpha}\,\bar{y}$. Hence $\bar{x} \in \Lambda(X)$ and this completes the proof.

Note : It clearly follows from the above result that a normal VVSS is always monotone.

The main result of this section which connects the α-dual of a VVSS $\Lambda(X)$ with its β-, γ-, and δ-duals, is the following

THEOREM 3.2 : For a dual pair $\langle X,Y\rangle$, we have

(i) $\Lambda^{\times}(Y) = \Lambda^{\beta}(Y)$ if $\Lambda(X)$ is monotone.

(ii) $\Lambda^{\times}(Y) = \Lambda^{\gamma}(Y)$ if $\Lambda(X)$ is normal.

(iii) $\Lambda^{\times}(Y) = \Lambda^{\delta}(Y)$ if $\Lambda(X)$ is symmetric.

PROOF : (i) Since $\Lambda^{\times}(Y) \subset \Lambda^{\beta}(Y)$, it suffices to show that $\Lambda^{\beta}(Y) \subset \Lambda^{\times}(Y)$. Consider $\{y_i\} \in \Lambda^{\beta}(Y)$. Since $m_o \cdot \Lambda(X) \subset \Lambda(X)$, it follows that $\sum_{i\geq 1} \langle \alpha_i x_i, y_i\rangle$ converges for every $\{\alpha_i\} \in m_o$ and $\{x_i\} \in \Lambda(X)$. Thus the series $\sum_{i\geq 1} \langle x_i, y_i\rangle$ is subseries convergent in $\mathbb{K}$ and hence $\sum_{i\geq 1} |\langle x_i,y_i\rangle| < \infty$ (cf. [75], Theorem 2, p. 22). Consequently, $\{y_i\} \in \Lambda^{\times}(Y)$.

(ii) Again, it is sufficient to prove here that $\Lambda^{\gamma}(Y) \subset \Lambda^{\times}(Y)$, for the other inclusion is always true. Let $\{y_i\}$ be a member of $\Lambda^{\gamma}(Y)$. For $\{x_i\}$ in $\Lambda(X)$, choose $\{\alpha_i\}$ in $\mathbb{K}$ such that $|\langle x_i,y_i\rangle| = \alpha_i \langle x_i,y_i\rangle$, for each $i \geq 1$. Then $|\alpha_i| = 1$ and so $\{\alpha_i x_i\} \in \Lambda(X)$. Moreover,

$$\sup_n \sum_{i=1}^{n} |\langle x_i,y_i\rangle| = \sup_n \left|\sum_{i=1}^{n} \langle \alpha_i x_i, y_i\rangle\right| < \infty$$

$$\Longrightarrow \sum_{i=1}^{n} |\langle x_i,y_i\rangle| \leq M, \quad \forall\, n \geq 1$$

where $M = \sup_n \sum_{i=1}^{n} |\langle x_i,y_i\rangle|$. Since n is arbitrary, we conclude $\sum_{i=1}^{\infty} |\langle x_i,y_i\rangle| < \infty$. This is true for any $\{x_i\} \in \Lambda(X)$ and so $\{y_i\} \in \Lambda^{\times}(Y)$.

(iii) The equality will follow if we show that $\Lambda^{\times}(Y) \subset \Lambda^{\delta}(Y)$. Consider, therefore, $\{y_i\}$ in $\Lambda^{\times}(Y)$. For any permutation π of $\mathbb{N}$, we have

$$\bar{x}_{\pi^{-1}} = \{x_{\pi^{-1}(i)}\} \in \Lambda(X)$$

for $\{x_i\}$ in $\Lambda(X)$, as $\Lambda(X)$ is symmetric. Therefore

$$\sum_{i \geq 1} |\langle x_i, y_{\pi(i)} \rangle| = \sum_{i \geq 1} |\langle x_{\pi^{-1}(i)}, y_i \rangle| < \infty.$$

Hence $\{y_i\} \in \Lambda^{\delta}(Y)$. Thus the proof is completely established.

4. TOPOLOGICAL DUALS OF $\Lambda(X)$

This section is devoted to the sequential representation of the members of the topological dual of a VVSS $\Lambda(X)$ equipped with a Hausdorff locally convex topology $\mathcal{T}$. For a VVSS $(\Lambda(X), \mathcal{T})$, we write

$$[\Lambda(X)]_c = \overline{\Phi(X)}$$

and

$$[\Lambda(X)]_s = \{\{f \circ R_i\} : f \in [\Lambda(X)]^*\},$$

where $[\Lambda(X)]^*$ stands for the topological dual of $(\Lambda(X), \mathcal{T})$. Since $f \circ R_i \in X'$ for each $i \geq 1$, we have $[\Lambda(X)]_s \subset \Omega(X')$. We begin with

PROPOSITION 4.1 : Let X be a vector space and $\Lambda(X)$ be equipped with a Hausdorff locally convex topology $\mathcal{T}$. Then

$$[\Lambda(X)]_s = [[\Lambda(X)]_c]_s \equiv [\overline{\Phi(X)}]_s .$$

PROOF : Since $[\Lambda(X)]_c \subset \Lambda(X)$, we have $[\Lambda(X)]^* \subset [\Lambda(X)]_c^*$ and therefore it follows that

$$[\Lambda(X)]_s \subset [[\Lambda(X)]_c]_s .$$

For the other inclusion, consider $\{f \circ R_i\}$ in $[[\Lambda(X)]_c]_s$ where $f \in [\Lambda(X)]_c^*$. Applying Proposition 1.5, Chapter 2, we get $g \in [\Lambda(X)]^*$ such that $g(\bar{x}) = f(\bar{x})$ for all $\bar{x}$ in $[\Lambda(X)]_c$. Then $\{g \circ R_i\} \in [\Lambda(X)]_s$ and $(g \circ R_i)(x) = g(\delta_i^x)$, $x \in X$. But $g(\delta_i^x) = f(\delta_i^x)$, for all $x \in X$. Hence $g \circ R_i = f \circ R_i$, $i \geq 1$. Hence the result follows.

There is another way of representing $[\Lambda(X)]_s$, for which we need introduce a few more notations for our convenience. Let us consider the dual pair $\langle \Phi(X), \Omega(X')\rangle$, where X is a vector space. Then for $A \subset \Phi(X)$, set

$$A^{\Omega(X')} = \{\bar{y} \in \Omega(X') : |\langle \bar{x}, \bar{y}\rangle| \leq 1, \text{ for all } \bar{x} \in A\}.$$

For any semi-norm p on $\Phi(X)$, write

$$A_p = \{\bar{x} \in \Phi(X) : p(\bar{x}) \leq 1\} .$$

[Indeed, $A^{\Omega(X')}$ is nothing but the polar of A in $\Omega(X')$ and A_p is the closed unit ball in $\Phi(X)$ corresponding to the semi-norm p].

We now have the desired result in

PROPOSITION 4.2 : For a VVSS $\Lambda(X)$ equipped with a locally

convex topology $\mathcal{J}$ generated by the family D_Λ of semi-norms,

$$[\Lambda(X)]_s = \bigcup\{A_p^{\Omega(X')}: p \in D_\Lambda\}$$

where

$$A_p^{\Omega(X')} \equiv (A_p)^{\Omega(X')}.$$

PROOF : Let $\bar{f} \in \bigcup\{A_p^{\Omega(X')}: p \in D_\Lambda\}$. Then $\bar{f} \in A_p^{\Omega(X')}$ for some $p \in D_\Lambda$ and $\bar{f} = \{f_i\}$, $f_i \in X'$ for $i \geq 1$. Also, by definition of $A_p^{\Omega(X')}$, we have

$$p(\bar{x}) = \sup \{|\langle\bar{x},\bar{f}\rangle| : \bar{f} \in A_p^{\Omega(X')}\} .$$

Hence

$$|\langle\bar{x},\bar{f}\rangle| \leq p(\bar{x}), \quad \forall\ \bar{x} \in \Phi(X).$$

Define $G_{\bar{f}}: \Phi(X) \to \mathbb{K}$ by $G_{\bar{f}}(\bar{x}) = \langle\bar{x},\bar{f}\rangle$. Then

$$|G_{\bar{f}}(\bar{x})| \leq p(\bar{x}), \ \forall\ \bar{x} \in \Phi(X),$$

and so $G_{\bar{f}} \in [\Phi(X)]^*$. Let $\hat{G}_{\bar{f}}$ be the unique extension of $G_{\bar{f}}$ to $\overline{\Phi(X)}$. Then $\{\hat{G}_{\bar{f}} \circ R_i\} \in [[\Phi(X)]_c]_s = [\Lambda(X)]_s$ by Proposition 4.1. Now for $x \in X$,

$$\begin{aligned}(\hat{G}_{\bar{f}} \circ R_i)(x) &= \hat{G}_{\bar{f}}(\delta_i^x) \\ &= \langle\delta_i^x,\bar{f}\rangle = \langle x,f_i\rangle\end{aligned}$$

$$\Longrightarrow \quad \hat{G}_{\bar{f}} \circ R_i = f_i, \text{ for all } i \geq 1.$$

Hence $\bar{f} \in [\Lambda(X)]_s$ and therefore, $\bigcup\{A_p^{\Omega(X')} : p \in D_\Lambda\}$ is contained in $[\Lambda(X)]_s$.

For the other inclusion, consider $\bar{f} \in [\Lambda(X)]_s$. Then $\bar{f} = \{f \circ R_i\}$ where $f \in [\Lambda(X)]^*$. There exists $p \in D_\Lambda$ such that

$$|\langle \bar{x}, f\rangle| \leq p(\bar{x}), \ \forall\ \bar{x} \in \Lambda(X).$$

Now for $\bar{x} \in \Phi(X)$, we have $\bar{x} = \sum_{i=1}^{n} \delta_i^{x_i}$, for some $n \in \mathbb{N}$. Therefore

$$\langle \bar{x}, f\rangle = f\Big(\sum_{i=1}^{n} \delta_i^{x_i}\Big) = \sum_{i=1}^{n} (f \circ R_i)(x_i)$$

$$= \langle \bar{x}, \bar{f}\rangle$$

$$\Longrightarrow \qquad |\langle \bar{x}, \bar{f}\rangle| \leq p(\bar{x}), \ \forall\ \bar{x} \in \Phi(X)$$

Since $p(\bar{x}) \leq 1$, for all $\bar{x} \in A_p$ we have $|\langle \bar{x}, \bar{f}\rangle| \leq 1$ for all $\bar{x} \in A_p$ and so $\bar{f} \in A_p^{\Omega(X')}$. Consequently,

$$[\Lambda(X)]_s \subset \bigcup\{A_p^{\Omega(X')} : p \in D_\Lambda\}.$$

Hence the result follows.

Next, we have

PROPOSITION 4.3 : For a VVSS $(\Lambda(X), \mathcal{F})$ where X is a vector space, $[\Lambda(X)]_c^*$ is algebraically isomorphic to the space $[[\Lambda(X)]_c]_s$.

PROOF : Define a mapping $\psi : [\Lambda(X)]_c^* \to [[\Lambda(X)]_c]_s$ by

$$\psi(f) = \{f \circ R_i\} \ , \ f \in [\Lambda(X)]_c^* \ .$$

Then ψ is linear ; for, if $f_1, f_2 \in [\Lambda(X)]_c^*$,

$\psi(f_1+f_2) = \{(f_1+f_2) \circ R_i\} = \{f_1 \circ R_i\} + \{f_2 \circ R_i\} = \psi(f_1) + \psi(f_2)$,

and if $\alpha \in \mathbb{K}$, $f \in [\Lambda(X)]_c^*$, $\psi(\alpha f) = \{(\alpha f) \circ R_i\} =$

$\alpha \{f \circ R_i\} = \alpha \ \psi(f)$.

Now

$$\begin{aligned} \psi(f) = 0 \ &\Longrightarrow \ \{f \circ R_i\} = 0 \\ &\Longrightarrow \ f \circ R_i = 0, \text{ for all } i \geq 1 \\ &\Longrightarrow \ (f \circ R_i)(x) = 0, \text{ for all } x \in X, \ i \geq 1 \\ &\Longrightarrow \ f(\delta_i^x) = 0 \ , \text{ for all } x \in X, \ i \geq 1 \\ &\Longrightarrow \ f(y) = 0 \ \text{ for all } \ y \in \Phi(X) \\ &\Longrightarrow \ f(z) = 0 \ \text{ for all } \ z \in \overline{\Phi(X)} = [\Lambda(X)]_c \\ &\Longrightarrow \ f = 0. \end{aligned}$$

Thus ψ is an injective map. It is clearly onto. Hence ψ is an algebraic isomorphism from $[\Lambda(X)]_c^*$ onto $[[\Lambda(X)]_c]_s$ and the result follows.

This result immediately leads to

COROLLARY 4.4 : If $(\Lambda(X), \mathfrak{J})$ is a GAD-space, then $[\Lambda(X)]^*$ is algebraically isomorphic to $[\Lambda(X)]_s$.

PROOF : Since $(\Lambda(X), \mathfrak{J})$ is a GAD-space, it follows from definition that $[\Lambda(X)]_c = \Lambda(X)$. Now, by the above result $[\Lambda(X)]_c^* = [\Lambda(X)]^*$ is algebraically isomorphic to the space $[[\Lambda(X)]_c]_s = [\Lambda(X)]_s$.

Note : We observe that for a GAD-space $\Lambda(X)$, each member f of its topological dual can be identified with a sequence $\{f \circ R_i\}$ in $[\Lambda(X)]_s$ and vice-versa. In other words, we can treat the two spaces as the same under this identification. This identification will turn out to be very useful in proving the results of this section.

PROPOSITION 4.5 : If $\Lambda(X)$ is a GAK-space, then $[\Lambda(X)]^* \subset \Lambda^\beta(X')$.

PROOF : Let us consider a member f of $[\Lambda(X)]^*$. The space $\Lambda(X)$, being a GAK-space, is also a GAD-space and so from Corollary 4.4 it follows that $f \equiv \{f \circ R_i\}$. Consider $\bar{x} \in \Lambda(X)$. From the hypothesis, $\bar{x}^{(n)} \to \bar{x}$, as $n \to \infty$ in $\mathcal{J}$ and therefore $f(\bar{x}^{(n)}) \to f(\bar{x})$, as $n \to \infty$. But

$$f(\bar{x}^{(n)}) = \sum_{i=1}^{n} \langle x_i, f \circ R_i \rangle .$$

Hence

$$f(\bar{x}) = \sum_{i \geq 1} \langle x_i, f \circ R_i \rangle .$$

Since $\bar{x} \in \Lambda(X)$ is arbitrary, $\{f \circ R_i\} \in \Lambda^\beta(X')$, and this completes the proof.

COROLLARY 4.6 : If X is a locally convex space and $(\Lambda(X), \mathcal{J})$ is a GAK and GC-space, then $[\Lambda(X)]^*$ is contained in $\Lambda^\beta(X^*)$.

PROOF : As the space $\Lambda(X)$ is a GC-space, each R_i is continuous and so $f \in [\Lambda(X)]^*$, implies that $f \circ R_i \in X^*$ for each $i \geq 1$. The result now follows from Proposition 4.5.

Restricting $(\Lambda(X), \mathcal{J})$ further, we get

PROPOSITION 4.7 : If $(\Lambda(X), \mathcal{F})$ is a barrelled, GAK- and GC-space, then $[\Lambda(X)]^* = \Lambda^\beta(X^*)$.

PROOF : In view of the Corollary 4.7, we need only show that $\Lambda^\beta(X^*) \subset [\Lambda(X)]^*$. Let $\bar{f} = \{f_i\} \in \Lambda^\beta(X^*)$. Then for $\bar{x} \in \Lambda(X)$, $\sum_{i \geq 1} \langle x_i, f_i \rangle$ converges. Define a linear functional $F: \Lambda(X) \to \mathbb{K}$, by

$$F(\bar{x}) = \sum_{i \geq 1} \langle x_i, f_i \rangle .$$

For $n \geq 1$, write

$$F_n(\bar{x}) = \sum_{i=1}^{n} \langle x_i, f_i \rangle$$

Then the sequence $\{F_n\} \subset X^*$, for if $\{\bar{x}^\delta\}$ is a net in $\Lambda(X)$, that converges to 0 in $\mathcal{F}$, $x_i^\delta = P_i(\bar{x}^\delta) \to 0$, for all $i \geq 1$, as $\Lambda(X)$ is a GK-space; and therefore $f_i(x_i^\delta) \to 0$, for each $i \geq 1$, which in turn, implies $\sum_{i=1}^{n} \langle x_i^\delta, f_i \rangle \to 0$, for all $n \geq 1$. Thus $F_n(\bar{x}^\delta) \to 0$, for all $n \geq 1$. Clearly $F_n(\bar{x}) \to F(\bar{x})$, as $n \to \infty$, for all $\bar{x} \in \Lambda(X)$ and $\{F_n\}$ is pointwise bounded. Therefore by Proposition 2.11 of Chapter 2, $F \in [\Lambda(X)]^*$. Consequently, $F \equiv \{F \circ R_i\}$, Now for $x \in X$, and $i \geq 1$

$$(F \circ R_i)(x) = F(\delta_i^x) = \langle x, f_i \rangle$$

$$\Longrightarrow F \circ R_i = f_i \ , \text{ for all } \ i \geq 1.$$

Thus, $\bar{f} = \{f_i\} \equiv \{F \circ R_i\} \in [\Lambda(X)]^*$ and hence the result follows.

COROLLARY 4.8 : Let $(\Lambda(X), \mathfrak{J})$ be a barrelled GAK- and GC-space. Then

(i) $[\Lambda(X)]^* = \Lambda^{\times}(X^*)$ if $\Lambda(X)$ is monotone

(ii) $[\Lambda(X)]^* = \Lambda^{\gamma}(X^*)$ if $\Lambda(X)$ is normal

PROOF : (i) When $\Lambda(X)$ is monotone, $\Lambda^{\beta}(X^*) = \Lambda^{\times}(X^*)$ by Theorem 3.2 and so (i) holds by the above proposition.

(ii) Since every normal space is also monotone, once again applying Theorem 3.2 and the above proposition, the result follows.

5. SEQUENTIAL DUALS OF $\Lambda(X)$

In this section we deal with the duality of a VVSS $\Lambda(X)$ corresponding to a dual pair $\langle X, X^+\rangle$, where X^+ is the sequential dual of a locally convex space X. Indeed, if $\Phi(X) \subset \Lambda(X)$ and

$$\Lambda^{\beta}(X^+) = \{\{f_i\}: f_i \in X^+,\ i \geq 1 \text{ and } \sum_{i\geq 1} \langle x_i, f_i\rangle \text{ converges}$$
$$\text{for all } \bar{x} \in \Lambda(X)\},$$

the pair $\langle \Lambda(X), \Lambda^{\beta}(X^+)\rangle$ forms a dual system. For a topological VVSS $(\Lambda(X), \mathfrak{J})$, we denote its sequential dual by $[\Lambda(X)]^+$. Analogous to the space $[\Lambda(X)]_s$ of the previous section, let us introduce the subspace $[\Lambda(X)]_{s(+)}$ of $\Omega(X^+)$ given by

$$[\Lambda(X)]_{s(+)} = \{\{f \circ R_i\} : f \in [\Lambda(X)]^+\} .$$

Then, concerning the dual $[\Lambda(X)]^+$, we have

PROPOSITION 5.1 : If $(\Lambda(X),\mathcal{F})$ is a GAK-space, $[\Lambda(X)]^+$ is algebraically isomorphic to the space $[\Lambda(X)]_{s(+)}$.

PROOF : Define a mapping $\psi : [\Lambda(X)]^+ \to [\Lambda(X)]_{s(+)}$ by

$$\psi(f) = \{f \circ R_i\} , \; f \in [\Lambda(X)]^+$$

Clearly ψ is a linear surjective map. Also

$$\begin{aligned} \psi(f) = 0 &\implies f \circ R_i = 0, \text{ for all } i \geq 1 \\ &\implies (f \circ R_i)(x) = 0, \text{ for all } x \in X, \; i \geq 1 \\ &\implies f(\delta_i^x) = 0, \text{ for all } x \in X, \; i \geq 1. \end{aligned}$$

Since $\Lambda(X)$ is a GAK-space and $f \in [\Lambda(X)]^+$, we have for $\bar{x} \in \Lambda(X)$, $\bar{x}^{(n)} \to \bar{x}$ in $\mathcal{F}$ and so $f(\bar{x}^{(n)}) \to f(\bar{x})$ as $n \to \infty$. As $f(\bar{x}^{(n)}) = \sum_{i=1}^{n} f(\delta_i^{x_i}) = 0$, for all $n \geq 1$, it follows that $f(\bar{x}) = 0$. Hence $f \equiv 0$. Thus the two spaces $[\Lambda(X)]^+$ and $[\Lambda(X)]_{s(+)}$ are algebraically isomorphic.

Note : The correspondence between $[\Lambda(X)]_{s(+)}$ and $[\Lambda(X)]^+$ established in Proposition 5.1 will throughout be assumed in the rest of this section without further reference.

PROPOSITION 5.2 : For a GAK- and GSC-space $(\Lambda(X),\mathcal{F})$, $[\Lambda(X)]^+ \subset \Lambda^\beta(X^+)$.

PROOF : For $f \in [\Lambda(X)]^+$, we have $f \equiv \{f \circ R_i\}$, by Proposition 5.1. As the space $\Lambda(X)$ is a GSC-space, $\{f \circ R_i\} \in \Omega(X^+)$. Also, $\bar{x}^{(n)} \to \bar{x}$ as $n \to \infty$, for all $\bar{x}$ in $\Lambda(X)$, implies that $f(\bar{x}^{(n)}) \to f(\bar{x})$, as $n \to \infty$. Since $f(\bar{x}^{(n)}) = \sum_{i=1}^{n} \langle x_i, f \circ R_i \rangle$, it follows that

$$f(\bar{x}) = \sum_{i \geq 1} \langle x_i, f \circ R_i \rangle .$$

Hence $f \equiv \{f \circ R_i\} \in \Lambda^\beta(X^+)$. This completes the proof.

For showing the equality of $[\Lambda(X)]^+$ and $\Lambda^\beta(X^+)$, we need prove the following two results.

LEMMA 5.3 : Let (X,T) be an l.c. TVS and f be the $\sigma(X^+,X)$-limit of a sequence $\{f_n\}$ in X^+. Then $f \in X'$.

PROOF : Given $x,y \in X$ and $\epsilon > 0$, there exists $n_o \in \mathbb{N}$ such that for $n \geq n_o$

$$|f_n(x) - f(x)| < \epsilon/3; \ |f_n(y) - f(y)| < \epsilon/3 ;$$

$$\text{and} \quad |f_n(x+y) - f(x+y)| < \epsilon/3 .$$

Then the above inequalities imply that

$$|f(x+y) - f(x) - f(y)| < \epsilon$$

which clearly proves the additive nature of f.
Similarly, we can prove the homogeneous character of f.
Thus $f \in X'$.

LEMMA 5.4 : Let (X,T) be an l.c. TVS such that every $\sigma(X^+,X)$-bounded sequence in X^+ is T-limited. If f is the $\sigma(X^+,X)$-limit of a sequence $\{f_n\}$ in X^+, then $f \in X^+$.

PROOF : In view of Lemma 5.3, we need show the sequential continuity of f. So, let $\{x_n\}$ be a null sequence in X. Since the sequence $\{f_n\}$ being $\sigma(X^+,X)$-bounded, is T-limited, to each $\epsilon > 0$ we can find $n_o \in \mathbb{N}$ such that for $n \geq n_o$

$$(*) \qquad |f_i(x_n)| < \epsilon/2, \ \forall\, i \geq 1.$$

Also for each $n \in \mathbb{N}$, there exists a positive integer k_o depending on n, such that

$$(**) \qquad |f_{k_o}(x_n) - f(x_n)| < \epsilon/2 .$$

Fixing $n \geq n_o$, the inequality

$$|f(x_n)| \leq |f(x_n) - f_{k_o}(x_n)| + |f_{k_o}(x_n)|$$

along with (*) and (**) implies that

$$|f(x_n)| < \epsilon, \quad \forall\, n \geq n_o.$$

Thus $f(x_n) \to 0$, as $n \to \infty$, and consequently $f \in X^+$.

We now come to the desired

PROPOSITION 5.5 : Let $(\Lambda(X),\mathcal{F})$ be a GAK-and GC-space such that every $\sigma([\Lambda(X)]^+, \Lambda(X))$-bounded sequence in $[\Lambda(X)]^+$ is $\mathcal{F}$-limited. Then

$$[\Lambda(X)]^+ = \Lambda^\beta(X^+).$$

PROOF : For $\bar{g} = \{g_i\} \in \Lambda^{\beta}(X^+)$, define linear maps F and F_n's from $\Lambda(X)$ to $\mathbb{K}$ by

$$(*) \qquad F(\bar{x}) = \sum_{i \geq 1} \langle x_i, g_i \rangle$$

and

$$F_n(\bar{x}) = \sum_{i=1}^{n} \langle x_i, g_i \rangle \; .$$

Proceeding on similar lines as in the proof of Proposition 4.7, we establish that $F_n \in [\Lambda(X)]^+$, for $n \geq 1$. From $F_n(\bar{x}) \to F(\bar{x})$ as $n \to \infty$ and the Lemma 5.4, $F \in [\Lambda(X)]^+$. Thus, we can write $F \equiv \{F \circ R_i\}$ by Proposition 5.1. Now from $(*)$ it is easy to verify that

$$F \circ R_i = g_i, \quad \forall \, i \geq 1.$$

Hence $\bar{g} \equiv \{F \circ R_i\} \equiv F \in [\Lambda(X)]_{s(+)}$. The result is now immediate from Proposition 5.2.

Concerning the relationship amongst various duals of a VVSS $\Lambda(X)$, one has the following

PROPOSITION 5.6 : Let (X,T) be a locally convex Mazur space and $\Lambda(X)$ be normal. Assume that $\Lambda(X)$, equipped with a locally convex topology $\mathcal{F}$, compatible with the dual pair $\langle \Lambda(X), \Lambda^{\times}(X^*) \rangle$, is a GSC-space. Then

$$[\Lambda(X)]^+ = [\bar{\Lambda}(X)]^* = \Lambda^{\times}(X^*).$$

Hence, $\mathcal{F}^+$ is also compatible with the dual pair $\langle \Lambda(X), \Lambda^{\times}(X^*) \rangle$.

PROOF : As $\mathfrak{F}$ is compatible with the dual pair $\langle\Lambda(X), \Lambda^{\times}(X^*)\rangle$ we have

$$(*) \qquad [\Lambda(X)]^* = \Lambda^{\times}(X^*)$$

and

$$(**) \qquad \sigma(\Lambda(X), \Lambda^{\times}(X^*)) \leq \mathfrak{F} \leq \tau(\Lambda(X), \Lambda^{\times}(X^*)),$$

(**) being true by Proposition 2.2, Chapter 2. Since the inclusion $[\Lambda(X)]^* \subset [\Lambda(X)]^+$ always holds, in view of (*) we need show that $[\Lambda(X)]^+ \subset \Lambda^{\times}(X^*)$. Now for $\bar{x} = \{x_i\} \in \Lambda(X)$, $\bar{x}^{(n)} \to \bar{x}$, as $n \to \infty$ in $\tau(\Lambda(X), \Lambda^{\times}(X^*))$ by Proposition 3.4, Chapter 2, and therefore, it follows from (**) that $(\Lambda(X), \mathfrak{F})$ is a GAK-space. An application of Proposition 5.2 yields

$$[\Lambda(X)]^+ \subset \Lambda^{\beta}(X^+).$$

Since X is a Mazur space and $\Lambda(X)$ is monotone, $\Lambda^{\beta}(X^+) = \Lambda^{\times}(X^*)$ and hence the result holds. Also, by Proposition 2.21, Chapter 2,

$$(\Lambda(X), \mathfrak{F}^+)^+ = (\Lambda(X), \mathfrak{F}^+)^* = (\Lambda(X), \mathfrak{F})^+.$$

Thus $\mathfrak{F}^+$ is compatible with the dual pair $\langle\Lambda(X), \Lambda^{\times}(X^*)\rangle$.

COROLLARY 5.7 : If (X,T) is a bornological space and $(\Lambda(X), \mathfrak{F})$ is a normal GSC-space such that $\mathfrak{F}$ is compatible with the dual pair $\langle\Lambda(X), \Lambda^{\times}(X^*)\rangle$, then

$$[\Lambda(X)]^+ = [\Lambda(X)]^* = \Lambda^{\times}(X^*).$$

PROOF : From Proposition 2.21, Chapter 2,

$$(X,T^+)^+ = (X,T^+)^* = (X,T)^+ \text{ and } T = T^+$$

Therefore $X^+ = X^*$ and the result is now immediate by the preceding proposition.

COROLLARY 5.8 : Let (X,T) be a locally convex Mazur space and $\Lambda(X)$ be normal. Then

$$[\sigma(\Lambda(X), \Lambda^\times(X^*))]^+ \leq \tau(\Lambda(X), \Lambda^\times(X^*)).$$

PROOF : We need show here the compatibility of $[\sigma(\Lambda(X),\Lambda^\times(X^*)]^+$ with the dual pair $\langle\Lambda(X),\Lambda^\times(X^*)\rangle$. To achieve this, it is sufficient to prove that $(\Lambda(X),\sigma(\Lambda(X),\Lambda^\times(X^*)))$ is a GSC-space in view of Proposition 5.6. Let us consider the maps $R_i : X \to \Lambda(X)$, $R_i(x) = \delta_i^x$, $x \in X$ and $i \geq 1$. For $\bar{f} = \{f_i\}$ in $\Lambda^\times(X^*)$, we have

$$p_{\bar{f}}(R_i(x)) = |\langle\delta_i^x, \bar{f}\rangle| = |\langle x,f_i\rangle| = q_{f_i}(x).$$

Hence the maps R_i's are $\sigma(X,X^*) - \sigma(\Lambda(X),\Lambda^\times(X^*))$ continuous and so $T - \sigma(\Lambda(X),\Lambda^\times(X^*))$ continuous. Thus $(\Lambda(X),\sigma(\Lambda(X),\Lambda^\times(X^*)))$ is a GSC-space.

COROLLARY 5.9 : Let (X,T) be a Mackey Mazur space and $\Lambda(X)$ be normal. Then

$$[\tau(\Lambda(X),\Lambda^\times(X^*))]^+ = \tau(\Lambda(X),\Lambda^\times(X^*)).$$

PROOF : As (X,T) is a Mackey Mazur space, $T = \tau(X,X^*)$ and $X^+ = X^*$. Once again, it suffices to show that $(\Lambda(X),\tau(\Lambda(X),\Lambda^\times(X^*)))$ is a GSC-space. Invoking the proof of Corollary 5.8, the maps R_i's are $\sigma(X,X^*) - \sigma(\Lambda(X),\Lambda^\times(X^*))$ continuous and so they are $\tau(X,X^*) - \tau(\Lambda(X),\Lambda^\times(X^*))$ continuous by Proposition 2.19, Chapter 2. Thus $(\Lambda(X), \tau(\Lambda(X),\Lambda^\times(X^*)))$ is a GC-space, and in particular, it is a GSC-space. Hence the result follows by Proposition 5.6 and Proposition 2.2, Chapter 2.

PROPOSITION 5.10 : Let (X,T) be a weakly complete, Mackey and Mazur space and $(\Lambda(X),\mathcal{F})$ a perfect GSC-, bornological space such that the convergent sequences in $\Lambda(X)$ relative to $\mathcal{F}$ and $\sigma(\Lambda(X), \Lambda^\times(X^*))$ are the same. Then

$$\mathcal{F} = \tau(\Lambda(X),\Lambda^\times(X^*)) = \beta(\Lambda(X),\Lambda^\times(X^*)).$$

PROOF : As convergent sequences in $\sigma(\Lambda(X),\Lambda^\times(X^*))$ and $\mathcal{F}$ are the same and $(\Lambda(X),\mathcal{F})$ is bornological, we have

$$\mathcal{F} = \mathcal{F}^+ = [\sigma(\Lambda(X),\Lambda^\times(X^*))]^+ \geq \sigma(\Lambda(X),\Lambda^\times(X^*))$$

Also, by Corollary 5.8,

$$\mathcal{F} = [\sigma(\Lambda(X),\Lambda^\times(X^*))]^+ \leq \tau(\Lambda(X),\Lambda^\times(X^*)).$$

Thus $\mathcal{F}$ is compatible with the dual pair $\langle\Lambda(X),\Lambda^\times(X^*)\rangle$, and therefore $\mathcal{F} = \tau(\Lambda(X),\Lambda^\times(X^*))$ by the bornological character of $(\Lambda(X),\mathcal{F})$. Now $(\Lambda(X), \eta(\Lambda(X),\Lambda^\times(X^*)))$ is complete by Theorem 2.3, Chapter 3 and $\eta(\Lambda(X),\Lambda^\times(X^*))$ is compatible with the dual pair

$\langle \Lambda(X), \Lambda^{\times}(X^*)\rangle$ (cf. Proposition 3.3, Chapter 2). Hence by Proposition 2.4, Chapter 2, $(\Lambda(X), \mathcal{J})$ is complete and therefore barrelled (cf. Proposition 2.12, Chapter 2). Consequently,

$$\mathcal{J} = \tau(\Lambda(X), \Lambda^{\times}(X^*)) = \beta(\Lambda(X), \Lambda^{\times}(X^*)).$$

6. $\sigma\mu$-TOPOLOGY

As mentioned in the beginning of this chapter, this section deals with the study of the μ-dual $\Lambda^{\mu}(Y)$ of a VVSS $\Lambda(X)$ and the $\sigma\mu$-topology on $\Lambda(X)$, corresponding to a dual pair $\langle X,Y\rangle$ of vector spaces X and Y and a topological SVSS (μ, T_{μ}).

The space (μ, T_{μ}), no doubt, carries a good deal of impact on $(\Lambda(X), \sigma\mu)$, and to begin with, we find a glimpse of this aspect contained in

PROPOSITION 6.1 : Let X be equipped with the topology $\sigma(X,Y)$ and (μ, T_{μ}) be a K-space. Then $(\Lambda(X), \sigma\mu)$ is a GK-space. Further, if (μ, T_{μ}) is an AK-space, then $(\Lambda(X), \sigma\mu)$ is a GAK-space.

PROOF : To show that $(\Lambda(X), \sigma\mu)$ is a GK-space, consider a net $\{\bar{x}^{\alpha} : \alpha \in \Delta\}$ in $\Lambda(X)$ such that $\bar{x}^{\alpha} \to 0$ relative to the $\sigma\mu$-topology. Thus for $\epsilon > 0$, $p \in D_{\mu}$ and $\bar{y} \in \Lambda^{\mu}(Y)$ we can find $\alpha_0 \in \Delta$, $\alpha_0 \equiv \alpha_0(\epsilon, p, \bar{y})$ such that

$$p(\{\langle x_i^{\alpha}, y_i\rangle\}) = p_{\bar{y}}(\bar{x}^{\alpha}) < \epsilon, \qquad \alpha \geq \alpha_0.$$

In particular, choose $\bar{y} = \delta_i^y$ for $y \in Y$ and $i \in \mathbb{N}$. Then for $\alpha \geq \alpha_0$

$$p_{\delta_i^y}(\bar{x}^{\alpha}) = p(\{0,0,0,\ldots,\underset{\substack{\downarrow \\ \text{i-th place}}}{\langle x_i^{\alpha},y\rangle},0,0,\ldots\})$$

$$= p(\langle x_i^{\alpha},y\rangle\, e^i) < \epsilon,$$

where e^i is the unit vector. Since (μ,T_μ) is a K-space it follows that $\langle x_i^{\alpha},y\rangle \to 0$, for each $i \geq 1$ and hence the maps $P_i : (\Lambda(X),\sigma\mu) \to (X,\sigma(X,Y))$, $i \geq 1$ are continuous. Thus $(\Lambda(X),\sigma\mu)$ is a K-space.

For proving the GAK-ness of $(\Lambda(X),\sigma\mu)$, consider an arbitrary $\bar{x}$ in $\Lambda(X)$. Then for $p \in D_\mu$ and $\bar{y} \in \Lambda^\mu(Y)$,

$$p_{\bar{y}}(\bar{x}^{(n)} - \bar{x}) = p(\{\langle x_i,y_i\rangle\}^{(n)} - \{\langle x_i,y_i\rangle\})$$

$$= p(\bar{\beta}^{(n)} - \bar{\beta}),$$

where $\bar{\beta} = \{\langle x_i,y_i\rangle\} \in \mu$. As (μ, T_μ) is an AK-space, $p(\bar{\beta}^{(n)} - \bar{\beta}) \to 0$ as $n \to \infty$, and the result follows.

Concerning the μ-perfectness of $\Lambda(X)$, we have

PROPOSITION 6.2 : Let X be equipped with the topology $\sigma(X,Y)$ and (μ,T_μ) be an AK-space. If $(\Lambda(X),\sigma\mu)$ is sequentially complete, then $\Lambda(X)$ is μ-perfect, that is, $\Lambda(X) = [\Lambda(X)]^{\mu\mu}$.

PROOF : To prove the result, we need show $[\Lambda(X)]^{\mu\mu} \subset \Lambda(X)$. Let $\bar{x} \in [\Lambda(X)]^{\mu\mu}$. Then $\{\bar{x}^{(n)} : n \geq 1\} \subset \Lambda(X)$. Consider

$\bar{y} \in \Lambda^{\mu}(Y)$ and $p \in D_{\mu}$. Write $\beta_i = \langle x_i, y_i \rangle$, $i \geq 1$. Then $\bar{\beta} = \{\beta_i\} \in \mu$ and for $m < n$, we have

$$p_{\bar{y}}(\bar{x}^{(n)} - \bar{x}^{(m)}) = p(\{0,0,\ldots,\langle x_{m+1}, y_{n+1}\rangle,\ldots,$$

$$, \langle x_n, y_n\rangle, 0, 0, \ldots\})$$

$$= p(\bar{\beta}^{(n)} - \bar{\beta}^{(m)}).$$

Since μ is an AK-space, the last term tends to 0 as $m,n \to \infty$. Hence $\{\bar{x}^{(n)}\}$ is a $\sigma\mu$-Cauchy sequence in $\Lambda(X)$. Therefore for a uniquely determined element $\bar{z}$ in $\Lambda(X)$,

$$\bar{x}^{(n)} \to \bar{z}, \text{ as } n \to \infty \text{ in } \sigma\mu .$$

Thus $x_i^{(n)} \to z_i$ as $n \to \infty$ for $i \geq 1$ in X relative to $\sigma(X,Y)$ topology (cf. Proposition 6.1). Since $(\bar{x}^{(n)})_i = x_i$, for all $n \geq i$, we have $x_i = z_i$ for all $i \geq 1$, and so $\bar{x} = \bar{z}$. Consequently, $\bar{x} \in \Lambda(X)$ and we conclude that $[\Lambda(X)]^{\mu\mu} \subset \Lambda(X)$. Therefore, $\Lambda(X)$ is μ-perfect.

Restricting X and (μ, T_{μ}) further, the converse of the above proposition is obtained in the following

PROPOSITION 6.3 : Let $(X, \sigma(X,Y))$ and (μ, T_{μ}) be (sequentially) complete such that (μ, T_{μ}) is also a K-space. If $\Lambda(X)$ is μ-perfect, then $\Lambda(X)$ is $\sigma\mu$-(sequentially) complete.

PROOF : We prove the result for completeness and similarly follows the other part for sequential completeness. Let us

therefore, consider a $\sigma\mu$-Cauchy net $\{\bar{x}^{\alpha} : \alpha \in \Delta\}$ in $\Lambda(X)$. Then from Proposition 6.1, $\{x_i^{\alpha}\}$ is a $\sigma(X,Y)$-Cauchy net in X, for each $i \geq 1$. Since $(X, \sigma(X,Y))$ is complete, there exists $\bar{x} \in \Omega(X)$ such that $x_i^{\alpha} \to x_i$ in $\sigma(X,Y)$, for each $i \geq 1$. In order to show the convergence of $\bar{x}^{\alpha}$ to $\bar{x}$ in $(\Lambda(X), \sigma\mu)$, let us take a point $\bar{y} \in \Lambda^{\mu}(Y)$. Then $\{\langle x_i^{\alpha}, y_i\rangle\}$ belongs to μ for each $\alpha \in \Delta$. Also the equality

$$p_{\bar{y}} = (\bar{x}^{\alpha} - \bar{x}^{\beta}) = p(\{\langle x_i^{\alpha}, y_i\rangle\} - \{\langle x_i^{\beta}, y_i\rangle\})$$

implies that $\{\langle x_i^{\alpha}, y_i\rangle\}$ is a Cauchy net in μ with respect to T_{μ}. Hence there exists a $\{\beta_i\} \in \mu$ such that,

$$\{\langle x_i^{\alpha}, y_i\rangle\} \overset{\alpha}{\to} \{\beta_i\} \text{ in } T_{\mu} .$$

Therefore,

$$\langle x_i^{\alpha}, y_i\rangle \overset{\alpha}{\to} \beta_i , \text{ for each } i \geq 1,$$

since (μ, T_{μ}) is a K-space. But

$$x_i^{\alpha} \overset{\alpha}{\to} x_i \text{ in } \sigma(X,Y) \text{ for each } i \geq 1$$

$$\Longrightarrow \quad \langle x_i^{\alpha}, y_i\rangle \to \langle x_i, y_i\rangle \text{ for all } i \geq 1.$$

Thus $\beta_i = \langle x_i, y_i\rangle$, for $i \geq 1$. Hence $\{\langle x_i, y_i\rangle\} \in \mu$. As $\bar{y} \in \Lambda^{\mu}(Y)$ is arbitrary, it follows that $\bar{x} \in [\Lambda(X)]^{\mu\mu} = \Lambda(X)$. But convergence of $\{\langle x_i^{\alpha}, y_i\rangle\}$ to $\{\beta_i\}$ in T_{μ} implies that $\bar{x}^{\alpha} \to \bar{x}$ in the space $(\Lambda(X), \sigma\mu)$. Therefore $(\Lambda(X), \sigma\mu)$ is complete.

Since the μ-dual of a VVSS $\Lambda(X)$ is always μ-perfect, the above result immediately leads to

COROLLARY 6.4 : Let (μ, T_μ) be a K-space. If $(Y, \sigma(Y,X))$ and (μ, T_μ) are (sequentially) complete, then the μ-dual $\Lambda^\mu(Y)$ of a VVSS $\Lambda(X)$, equipped with the topology $\sigma\mu(\Lambda^\mu(Y), \Lambda(X))$ is also (sequentially) complete.

PROOF : As mentioned above, $\Lambda^\mu(Y)$ is μ-perfect. Now by interchanging the roles of $\Lambda(X)$ and $\Lambda^\mu(Y)$ in the above proposition, we get the desired result.

Completely bounded sets

We observe that a set A in $\Lambda(X)$ is $\sigma\mu(\Lambda(X), \Lambda^\mu(Y))$ bounded if and only if the set $A\bar{y} = [\{\langle x_i, y_i \rangle\} : \bar{x} \in A]$ forms a bounded subset in μ for each $\bar{y} \in \Lambda^\mu(Y)$. Replacing the singletons $\{\bar{y}\}$ by a $\sigma\mu(\Lambda^\mu(Y), \Lambda(X))$-bounded subset of $\Lambda^\mu(Y)$, we introduce the following

DEFINITION 6.5 : A subset A of $\Lambda(X)$ is said to be <u>completely bounded</u> in $\Lambda(X)$ if $AB = [\{\langle x_i, y_i \rangle\} : \{x_i\} \in A \text{ and } \{y_i\} \in B]$ is a bounded subset of μ for each $\sigma\mu(\Lambda^\mu(Y), \Lambda(X))$-bounded subset B of $\Lambda^\mu(Y)$.

Note : Obviously every completely bounded subset of $\Lambda(X)$ is bounded for the $\sigma\mu$-topology on $\Lambda(X)$. For a restricted (μ, T_μ), the other implication also holds. Indeed, we have

THEOREM 6.6 : Let $\langle X,Y\rangle$ be a dual pair such that $(Y, \sigma(Y,X))$ is sequentially complete and (μ, T_μ) is a sequentially complete K-space. Then every $\sigma\mu(\Lambda(X), \Lambda^\mu(Y))$-bounded subset of $\Lambda(X)$ is completely bounded.

PROOF : Let us assume that there is a $\sigma\mu(\Lambda(X), \Lambda^\mu(Y))$-bounded subset A of $\Lambda(X)$ such that A is not completely bounded. Hence we can find a $\sigma\mu(\Lambda^\mu(Y), \Lambda(X))$-bounded set B in $\Lambda^\mu(Y)$ such that AB is not bounded in μ. Therefore, there exists a continuous semi-norm $p \in D_\mu$ such that $p(AB)$ is not bounded in $\mathbb{K}$. Thus, for given $\epsilon > 0$, we can find $\bar{x}^1 \in A$ and $\bar{y}^1 \in B$ with

$$p(\{\langle x_i^1, y_i^1\rangle\}) \geq 1 + \epsilon.$$

Since A and B are $\sigma\mu$-bounded in $\Lambda(X)$ and $\Lambda^\mu(Y)$ respectively, we can find constants K_1 and M_1 satisfying

$$\sup_{\bar{x}\in A} p_{\bar{y}^1}(\bar{x}) = \sup_{\bar{x}\in A} p(\{\langle x_i, y_i^1\rangle\}) \leq K_1 ;$$

and

$$\sup_{\bar{y}\in B} p_{\bar{x}^1}(\bar{y}) = \sup_{\bar{y}\in B} p(\{\langle x_i^1, y_i\rangle\}) \leq M_1.$$

Choose $m_1 \in \mathbb{N}$ such that $2^{-m_1+1} \leq \epsilon . M_1^{-1}$. Then from the unbounded character of $p(AB)$, for the constant $2^{m_1}(K_1 + 2 + \epsilon)$, we can find $\bar{x}^2 \in A$ and $\bar{y}^2 \in B$ such that

$$p(\{\langle x_i^2, y_i^2\rangle\}) \geq 2^{m_1}(K_1 + 2 + \epsilon).$$

From the $\sigma\mu$-boundedness of A and B, we can find constants K_2 and M_2 such that

$$\sup_{\bar{x}\in A} p_{\bar{y}^2}(\bar{x}) = \sup_{\bar{x}\in A} p(\{\langle x_i, y_i^2\rangle\}) \leq K_2;$$

and

$$\sup_{\bar{y}\in B} p_{\bar{x}^2}(\bar{y}) = \sup_{\bar{y}\in B} p(\{\langle x_i^2, y_i\rangle\}) \leq M_2.$$

Choose $m_2 > m_1$ so that $2^{-m_2+1} \leq \epsilon M_2^{-1}$. Then for the constant $2^{m_2}(K_1 + 2^{-m_1} K_2 + 3 + \epsilon)$, we can find $\bar{x}^3 \in A$ and $\bar{y}^3 \in B$ such that

$$p(\{\langle x_i^3, y_i^3\rangle\}) \geq 2^{m_2}(K_1 + 2^{-m_1} K_2 + 3 + \epsilon).$$

Once again the $\sigma\mu$-boundedness of A and B will enable us to find constants K_3 and M_3 such that

$$\sup_{\bar{x}\in A} p_{\bar{y}^3}(\bar{x}) = \sup_{\bar{x}\in A} p(\{\langle x_i, y_i^3\rangle\}) \leq K_3;$$

and

$$\sup_{\bar{y}\in B} p_{\bar{x}^3}(\bar{y}) = \sup_{\bar{y}\in B} p(\{\langle x_i^3, y_i\rangle\}) \leq M_3.$$

Choose $m_4 > m_3$ so that $2^{-m_3+1} \leq \epsilon M_3^{-1}$. Proceeding in this fashion, we get sequences $\{\bar{x}^n\}$ in A and $\{\bar{y}^n\}$ in B, $\{K_n\}$ and $\{M_n\}$ of positive constants and a subsequence $\{m_n\}$ of integers with $m_{n+1} > m_n$, $m_o = 0$ such that for $n = 1,2,3,\ldots$

$$p(\{\langle x_i^n, y_i^n\rangle\}) > 2^{m_{n-1}}\left(\sum_{i=0}^{n-1} 2^{-m_{i-1}} K_i + n + \epsilon\right),\ K_0 = 0;$$

$$\sup_{\bar{x}\in A} p_{\bar{y}^n}(\bar{x}) = \sup_{\bar{x}\in A} p(\{\langle x_i, y_i^n\rangle\}) \leq K_n;$$

$$\sup_{\bar{y}\in B} p_{\bar{x}^n}(\bar{y}) = \sup_{\bar{y}\in B} p(\{\langle x_i^n, y_i\rangle\}) \leq M_n;$$

and

$$2^{-m_n+1} \leq \epsilon\, M_n^{-1}\,.$$

Since $\{2^{-m_{j-1}}\} \in \ell^1$ and $\{\bar{y}^j\} \in B$, the sequence $\{\sum_{j=1}^{n} 2^{-m_{j-1}}\bar{y}^j\}$ forms a $\sigma\mu(\Lambda^\mu(Y),\Lambda(X))$-Cauchy sequence in $\Lambda^\mu(Y)$ (cf. Lemma 1.3, Chapter 2). Hence by Corollary 6.4, there exists a $\bar{z} \in \Lambda^\mu(Y)$ with

$$\bar{z} = \sum_{j\geq 1} 2^{-m_{j-1}}\bar{y}^j,$$

where the limit is taken in the $\sigma\mu(\Lambda^\mu(Y), \Lambda(X))$-topology.

For $n \in \mathbb{N}$, consider

$$\begin{aligned}
p_{\bar{z}}(\bar{x}^n) &= p\,(\{\langle x_i^n, z_i\rangle\}) \\
&\geq 2^{-m_{n-1}}\, p(\{\langle x_i^n, y_i^n\rangle\}) - p(\{\langle x_i^n, y_i^1\rangle\}) - \\
&\quad 2^{-m_1}\, p(\{\langle x_i^n, y_i^2\rangle\}) - \ldots - 2^{-m_{n-2}}\, p(\{\langle x_i^n, y_i^{n-1}\rangle\}) \\
&\quad - 2^{-m_n} M_n(1 + 2^{m_n - m_{n+1}} + 2^{m_n - m_{n+2}} + \ldots) \\
&\geq \left(\sum_{i=0}^{n-1} 2^{m_{i-1}} K_i + n + \epsilon\right) - K_1 - 2^{-m_1} K_2 - \ldots - 2^{-m_{n-2}} K_{n-1} - \epsilon \\
&= n\,.
\end{aligned}$$

As $\{\bar{x}^n\}$ is a subset of A, the above inequality contradicts the $\sigma\mu$-boundedness of A. Hence the result follows.

Corresponding to a dual system $\langle X,Y\rangle$ of vector spaces, it is not always true that the $\sigma(X,Y)$ and $\beta(X,Y)$-bounded sets are the same; however the situation becomes pleasant when these two types of boundedness coincide (for instance, see [44]). Accordingly we have from [57]

DEFINITION 6.7 : A dual system $\langle X,Y\rangle$ is said to be an <u>M-system</u> if each $\sigma(X,Y)$-bounded set is $\beta(X,Y)$-bounded in X, or equivalently, each $\sigma(Y,X)$-bounded set of Y is $\beta(Y,X)$-bounded.

If a VVSS $\Lambda(X)$ is normal, the $\sigma(\Lambda^\times(Y), \Lambda(X))$-bounded and $\eta(\Lambda^\times(Y), \Lambda(X))$-bounded sets in $\Lambda^\times(Y)$ are the same by Proposition 3.5, Chapter 2, however, for the dual pair $\langle\Lambda(X), \Lambda^\times(Y)\rangle$, we have

PROPOSITION 6.8 : Let $\langle X,Y\rangle$ be a dual pair such that $(Y, \sigma(Y,X))$ is sequentially complete. If $\Lambda(X)$ is normal, then every $\eta(\Lambda(X), \Lambda^\times(Y))$-bounded set is $\beta(\Lambda(X), \Lambda^\times(Y))$-bounded; in particular $\langle\Lambda(X), \Lambda^\times(Y)\rangle$ is an M-system.

PROOF : Restricting $\mu = \ell^1$ in Definition 6.5, we observe that the completely bounded sets in a VVSS $\Lambda(X)$ are nothing but the $\beta(\Lambda(X), \Lambda^\times(Y))$-bounded. Thus for $\mu = \ell^1$, it follows from Theorem 6.6 that every $\eta(\Lambda(X), \Lambda^\times(Y))$-bounded subset of $\Lambda(X)$ is $\beta(\Lambda(X), \Lambda^\times(Y))$-bounded. Hence $\langle\Lambda(X), \Lambda^\times(Y)\rangle$ is also an M-system. This completes the proof.

COMPACT SETS IN $\Lambda(X)$

1 INTRODUCTION

In this chapter, following [70] and [93] we characterize compact sets and normal hulls of bounded and relatively compact sets in a VVSS. Indeed, the results proved here are interesting generalizations of scalar valued analogues.

2. CHARACTERIZATION OF COMPACT SETS

Recalling the terminology of the foregoing chapter, we start with

PROPOSITION 2.1 : Let (X,T) be a Hausdorff l.c. TVS and $(\Lambda(X),\mathcal{F})$, equipped with a locally convex topology $\mathcal{F}$, be a GK-space. Then a subset A of $\Lambda(X)$ is relatively compact if and only if

(i) $P_i(A)$ is relatively compact in X, for each $i\geq 1$; and

(ii) if $\{\bar{x}^\alpha : \alpha \in \Delta\}$ is a net in A such that $x_i^\alpha \xrightarrow{\alpha} x_i$, for each $i\geq 1$ in X, relative to the topology T, then $\bar{x}^\alpha \to \bar{x}$, in $(\Lambda(X),\mathcal{F})$, where $\bar{x} = \{x_i\}$.

PROOF : Let us first assume that A is a relatively compact subset of $\Lambda(X)$. Then from the GK-character of $\Lambda(X)$ and the equality $P_i(\bar{A}) = \overline{P_i(A)}$ for $i\geq 1$, the relative compactness of each set $P_i(A)$ in X follows immediately. Thus (i) holds.

For the proof of part (ii), let $\{\bar{x}^{\alpha} : \alpha \in \Delta\}$ be a net in A such that $x_i^{\alpha} \xrightarrow{\alpha} x_i$ in (X,T) for $i \geq 1$. Since A is relatively compact, there is an adherent point, say $\bar{y} = \{y_i\}$, of the net $\{\bar{x}^{\alpha} : \alpha \in \Delta\}$ in $\Lambda(X)$. Then the continuity of P_i's and use of Proposition 1.10, Chapter 2, yield that $y_i = x_i$, for all $i \geq 1$. Thus $\bar{x}$ is the only adherent point of the net $\{\bar{x}^{\alpha} : \alpha \in \Delta\}$. Hence from Proposition 1.11, Chapter 2, $\bar{x}^{\alpha} \xrightarrow{\alpha} \bar{x}$ in $\Lambda(X)$ and (ii) is proved.

Conversely, let (i) and (ii) hold and $F = \{F_{\alpha} : \alpha \in \Delta\}$ be an ultrafilter in A. From (i) and Proposition 1.9, Chapter 2, the ultrafilter $\{P_i(F_{\alpha}) : \alpha \in \Delta\}$ converges to some point x_i in X, for each $i \geq 1$. If $\{\bar{x}^{\alpha} : \alpha \in \Delta\}$ is the net corresponding to the ultrafilter $\{F_{\alpha} : \alpha \in \Delta\}$, then $\{P_i(\bar{x}_{\alpha}) = x_i^{\alpha} : \alpha \in \Delta\}$ is the net corresponding to the ultrafilter $\{P_i(F_{\alpha}) : \alpha \in \Delta\}$ in X, for each $i \geq 1$ and so from Proposition 1.9, Chapter 2, the net $\{x_i^{\alpha} : \alpha \in \Delta\}$ converges to $x_i \in X$, for all $i \geq 1$. Now using (ii) $\bar{x}^{\alpha} \xrightarrow{\alpha} \bar{x}$ in $\Lambda(X)$ and therefore the ultrafilter $\{F_{\alpha} : \alpha \in \Delta\}$ converges to $\bar{x}$ in $\Lambda(X)$ by Proposition 1.9, Chapter 2. Hence A is relatively compact and thus the proof is complete.

We now prove our main result in

THEOREM 2.2 : Let (X,T) be a metrizable locally convex space and $(\Lambda(X),\mathcal{F})$ a GK-space such that $\Phi(X) \subset \Lambda(X)$. Then for

a subset M of $\Lambda(X)$, the following are equivalent:

(i) M is compact.

(ii) M is countably compact.

(iii) M is sequentially compact.

(iv) $P_i(M)$ is sequentially compact in X, for $i \geq 1$ and for a sequence $\{\bar{x}^n\}$ in M such that $x_i^n \to x_i^o$ in X, $i \geq 1$, the sequence $\{\bar{x}^n\}$ converges to $\bar{x}^o = \{x_i^o\}$ as $n \to \infty$ and $\bar{x}^o \in M$.

PROOF : (i) ==> (ii) is clear from the Definition 1.8, Chapter 2.

(ii) ==> (iii). In order to show this implication, we first prove that each $P_i(M)$ is sequentially compact. Let us fix a positive integer i_o and consider a sequence $\{y_{i_o}^n\}$ in $P_{i_o}(M)$. Then there exists a sequence $\{\bar{y}^n\}$ in M such that $P_{i_o}(\bar{y}^n) = y_{i_o}^n$, for all $n \geq 1$. From the countable compactness of M, there exists an adherent point $\bar{y} \in M$ of the sequence $\{\bar{y}^n\}$ and a subnet $\{\bar{y}^{n(\alpha)} : \alpha \in \Delta\}$ of the sequence $\{\bar{y}^n\}$ such that $\{\bar{y}^{n(\alpha)} : \alpha \in \Delta\}$ converges to $\bar{y}$. As P_{i_o} is continuous, the subnet $\{y_{i_o}^{n(\alpha)} : \alpha \in \Delta\}$ in $P_{i_o}(M)$ converges to a point y_{i_o} of $P_{i_o}(M)$. Hence y_{i_o} is an adherent point of the sequence $\{y_{i_o}^n\}$ in $P_{i_o}(M)$ and therefore $P_{i_o}(M)$ is countably compact. Now the metrizability of X and the arbitrary fixation of i_o imply that $P_i(M)$ is sequentially compact for $i \geq 1$.

Consider now a sequence $\{\bar{x}^n\}$ in M. Then the sequential compactness of each $P_i(M)$ implies that the sequence $\{x_i^n\}$ in $P_i(M)$ has a convergent subsequence for all $i \geq 1$. Utilizing the diagonal process, we can extract a subsequence from the sequence $\{\bar{x}^n\}$, which we once again denote by $\{\bar{x}^n\}$, such that

$$(*) \qquad x_i^n \to x_i^o, \text{ as } n \to \infty, \text{ in } T$$

for some $x_i^o \in P_i(M)$ and $i \geq 1$. Also, there is an adherent point $\bar{z} = \{z_i\}$ of the subsequence $\{\bar{x}^n\}$. By Proposition 1.10, Chapter 2, we get that the point z_i is an adherent point of the sequence $\{x_i^n\}$ in $P_i(M)$, for each $i \geq 1$. In view of (*), we have $z_i = x_i^o$, for $i \geq 1$. Thus $\bar{x}^o = \{x_i\}$, is the only adherent of the subsequence $\{\bar{x}^n\}$ and so on applying Proposition 1.11 of Chapter 2, we infer the convergence of $\{\bar{x}^n\}$ to $\bar{x}^o$ in M. Hence M is sequentially compact.

(iii) ==> (iv). As sequentially compact sets are countably compact, invoking the proof of the implication (ii) ==> (iii), the first part of (iv) follows. For the second part, consider a sequence $\bar{x}^n = \{x_i^n\}$, $n \geq 1$ in M such that for $i \geq 1$,

$$(**) \qquad x_i^n \to x_i^o, \text{ as } n \to \infty \text{ in } P_i(M).$$

Suppose that the sequence $\{\bar{x}^n\}$ does not converge to $\bar{x}^o = \{x_i^o\}$ in M. Then there exists a neighbourhood u of $\bar{x}^o$ and a subsequence $\{\bar{x}^{n_k}\}$ of the sequence $\{\bar{x}^n\}$ such that $\bar{x}^{n_k}$ is not in u, for all $k \geq 1$. Now the sequential compactness of M

Consider the sequence $\{\bar{u}^n\}$ in $\Lambda(X)$ such that

$$\bar{u}^n = \{u_1^n, u_2^n, \ldots, u_n^n, 0, 0, \ldots\}$$

where $\{u_1^n, u_2^n, \ldots, u_n^n\}$ is an adherent point of the net F_n, $n \geq 1$. Obviously the sequence $\{u_i^n\}$ is contained in $P_i(M)$ for $i \geq 1$. By utilizing the sequential compactness of $P_i(M)$ and the diagonal process, we can extract a subsequence of the sequence $\{\bar{u}^n\}$ which we once again designate by $\{\bar{u}^n\}$, such that

$$u_i^n \to u_i^o \quad \text{as} \quad n \to \infty \quad \text{in} \quad X$$

for some $u_i^o \in P_i(M)$ and each $i \geq 1$. Also, corresponding to $\bar{u}^n$ in $\Lambda(X)$ and an arbitrary $\epsilon > 0$, there is an $\bar{x}^{\alpha(n)}$ in F for which

$$(\times) \qquad \rho(u_i^n, x_i^{\alpha(n)}) < \epsilon, \; i=1,2,3,\ldots,n.$$

From (***) and $(\times)$ one can easily establish the pointwise convergence of $\bar{x}^{\alpha(n)}$ to $\bar{u}^o$ and hence by our hypothesis (i) follows.

3. COMPACTNESS OF NORMAL HULLS

This section includes results which are related to the relative compactness of normal hulls of bounded and relatively compact subsets of a VVSS. Throughout this section we consider a dual system $\langle\Lambda(X), \Lambda^\times(Y)\rangle$ of VVSS corresponding to a dual pair $\langle X,Y\rangle$ of vector spaces X and Y. We equip $\Lambda(X)$ with a

solid topology $\mathcal{F}$ as defined in section 3 of Chapter 2. We begin with the following basic result.

PROPOSITION 3.1 : Let X be equipped with the weak topology $\sigma(X,Y)$. Then $(\Lambda(X),\mathcal{F})$ is a GK-space.

PROOF : Let us consider a y in Y and $\bar{x}$ in $\Lambda(X)$. Then the inequality

$$q_y(P_i(\bar{x}))= |\langle x_i ,y \rangle| \leq R_S(\bar{x})$$

holds for $S \in \mathcal{G}$, that contains δ_i^y, where $i \in \mathbb{N}$. Hence P_i's are continuous and the result follows.

Next, we have

PROPOSITION 3.2 : Let $(\Lambda(X),\mathcal{F})$ be a normal VVSS and X be equipped with the topology $\sigma(X,Y)$. If the normal hull $\hat{K}$ of a bounded subset K of $\Lambda(X)$ is relatively compact, then

(i) $P_n(\hat{K})$ is relatively compact in X, for each $n \geq 1$; and

(ii) for $\epsilon > 0$ and $S \in \mathcal{G}$, there exists a positive integer i_o such that

$$\sum_{n \geq i_o+1} |\langle x_n ,y_n \rangle| \leq \epsilon,$$

for every $\bar{x} = \{x_n\} \in K$ and $\bar{y} = \{y_n\}$ in S.

PROOF : In view of Proposition 3.1, P_n's are continuous and therefore (i) is immediate.

For proving (ii) let us assume the contrary. Then there exist an $\epsilon > 0$ and $S \in \mathcal{G}$ such that for each $i \geq 1$, there are sequences $\{\bar{x}^i\}$ in A and $\{\bar{y}^i\}$ in S with

$$(*) \qquad \sum_{n \geq i+1} |x_n^i, y_n^i\rangle| > \epsilon.$$

Choose scalars λ_n^i with $|\lambda_n^i| \leq 1$, $i,n \geq 1$, as follows

$$\lambda_n^i = \begin{cases} \dfrac{|\langle x_n^i, y_n^i\rangle|}{\langle x_n^i, y_n^i\rangle} & \text{for } n>i \text{ and } \langle x_n^i, y_n^i\rangle \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then from (*), we get

$$(**) \qquad \sum_{n \geq 1} \langle \lambda_n^i x_n^i, y_n^i\rangle > \epsilon .$$

For each $i \geq 1$ the sequence $\{\lambda_n^i x_n^i\}_{n \geq 1}$, contained in $\hat{K}$, converges coordinatewise to zero in X and hence by Proposition 2.1, this sequence must also converge to zero in $\Lambda(X)$. Therefore for the above $\epsilon > 0$ and $S \in \mathcal{G}$, there exists $i_0 \in \mathbb{N}$ such that for $i \geq i_0$,

$$\Big| \sum_{n \geq 1} \langle \lambda_n^i x_n^i, y_n\rangle \Big| < \epsilon,$$

for all $\bar{y} \in S$. This clearly contradicts (**). Hence (ii) is true and the proof is complete.

Remark : It is clear from the above proof that Proposition 3.2 will remain valid if we replace the topology $\mathcal{J}$ by $\sigma(\Lambda(X), \mu(Y))$,

where $\mu(Y) \subset \Lambda^{\times}(Y)$ such that $\langle\Lambda(X), \mu(Y)\rangle$ forms a dual pair and the collection $\mathcal{G}$ by the collection of singleton sets in $\mu(Y)$.

As a partial converse of the above result, we have

PROPOSITION 3.3 : Let X be endowed with the topology $\sigma(X,Y)$ and $\Lambda(X)$ be a perfect VVSS which is equipped with $\eta(\Lambda(X), \Lambda^{\times}(Y))$. Then the normal hull $\hat{K}$ of a bounded subset K of $\Lambda(X)$ is relatively compact provided the following conditions are true:

(i) $P_n(\hat{K})$ is relatively compact in X, for each $i \geq 1$.

(ii) For $\bar{y} = \{y_n\} \in \Lambda^{\times}(Y)$ and $\epsilon > 0$, there exists a positive integer i_o such that

$$\sum_{n \geq i_o+1} |\langle x_n, y_n\rangle| < \epsilon,$$

for all $\bar{x} = \{x_n\} \in K$.

PROOF : Let $\{\bar{x}^{\alpha} : \alpha \in \Delta\}$ be a net in $\hat{K}$ such that $x_n^{\alpha} \xrightarrow{\alpha} x_n$ in X relative to $\sigma(X,Y)$, for all $n \geq 1$. Choose $\bar{y} = \{y_n\} \in \Lambda^{\times}(Y)$ and $\epsilon > 0$. Then by (ii), there exists a positive integer i_o such that

$$(*) \qquad \sum_{n \geq i_o+1} |\langle x_n^{\alpha}, y_n\rangle| < \epsilon/3, \ \forall\, \alpha \in \Delta$$

$$\Longrightarrow \qquad \sum_{n=i_o+1}^{m} |\langle x_n^{\alpha}, y_n\rangle| < \epsilon/3 ,$$

for all $\alpha \in \Delta$ and any $m \in \mathbb{N}$ with $m > i_o+1$. Hence

$$\sum_{n=i_o+1}^{m} |\langle x_n, y_n\rangle| < \epsilon/3 \quad \forall\, m \in \mathbb{N} \text{ with } m > i_o+1.$$

Consequently, $\bar{x} = \{x_n\} \in \Lambda^{\times\times}(X) = \Lambda(X)$, by hypothesis. Also, for the above $\epsilon > 0$ we can find $\alpha_o \in \Delta$ such that

$$\sum_{n=1}^{i_o} |\langle x_n^{\alpha} - x_n, y_n\rangle| < \epsilon/3 \text{ , for all } \alpha \geq \alpha_o \text{ .}$$

Now

$$\sum_{n\geq 1} |\langle x_n^{\alpha} - x_n, y_n\rangle| \leq \sum_{n=1}^{i_o} |\langle x_n^{\alpha} - x_n, y_n\rangle| + \sum_{n\geq i_o+1} |\langle x_n^{\alpha}, y_n\rangle| + \sum_{n\geq i_o+1} |\langle x_n, y_n\rangle|$$

$$\Longrightarrow \sum_{n\geq 1} |\langle x_n^{\alpha} - x_n, y_n\rangle| \leq \epsilon, \text{ for } \alpha \geq \alpha_o \text{ .}$$

As $\bar{y} \in \Lambda^{\times}(Y)$ is arbitrary, $\bar{x}^{\alpha} \xrightarrow{\alpha} \bar{x}$ in $\Lambda(X)$ for the topology $\eta(\Lambda(X), \Lambda^{\times}(Y))$. Now applying Proposition 2.1 we infer the relative compactness of $\hat{K}$, thereby completing the proof.

Combining Propositions 3.2 and 3.3, we state the main result in the form of

THEOREM 3.4 : Let X be endowed with the topology $\sigma(X,Y)$ and $(\Lambda(X), \eta(\Lambda(X), \Lambda^{\times}(Y)))$ be a perfect VVSS. Then the normal hull $\hat{K}$ of a bounded subset K of $\Lambda(X)$ is relatively compact if and only if the following conditions are true:

(i) $P_n(\hat{K})$ is relatively compact in X, for each $n \geq 1$.

(ii) For $\epsilon > 0$ and $\bar{y} \in \Lambda^\times(Y)$ there exists a positive integer i_o such that

$$\sum_{n \geq i_o+1} |\langle x_n, y_n \rangle| < \epsilon,$$

for all $\bar{x} \in K$.

Concerning the normal hull of a relatively compact set, we prove

PROPOSITION 3.5 : Let $(\Lambda(X), \mathfrak{F})$ be a normal sequence space such that $\mathfrak{F}$ is compatible with the dual pair $\langle \Lambda(X), \Lambda^\times(Y) \rangle$. Assume that the space Y is endowed with the topology $\sigma(Y,X)$. Then the normal hull $\hat{K}$ of a relatively compact subset K of $\Lambda(X)$ is relatively compact.

PROOF : Let A be a compact subset of ω(space of all scalar sequences) formed by the product of closed unit ball $\{\alpha : \alpha \in \mathbb{K}, |\alpha| \leq 1\}$ in $\mathbb{K}$. Define a map $f : A \times \Lambda(X) \to \Lambda(X)$ by

$$f(\bar{\alpha}, \bar{x}) = \bar{\alpha}\,\bar{x} = \{\alpha_i\, x_i\} \quad ,$$

where $\bar{\alpha} \in A$ and $\bar{x} \in \Lambda(X)$. Equip $A \times \Lambda(X)$ with the product topology. We now show that f is continuous.

Let us therefore, consider a net $\{z^\delta : \delta \in \Delta\}$, $z^\delta \equiv (\bar{\alpha}^\delta, \bar{x}^\delta)$ for all $\delta \in \Delta$, in $A \times \Lambda(X)$ such that $(\bar{\alpha}^\delta, \bar{x}^\delta) \xrightarrow{\delta} (\bar{\alpha}, \bar{x})$ in the product topology of $A \times \Lambda(X)$. Then $\bar{\alpha}^\delta \xrightarrow{\delta} \bar{\alpha}$ in A, and

$\bar{x}^{\delta} \xrightarrow{\delta} \bar{x}$ in $\Lambda(X)$ relative to the topology $\mathcal{F}$. Since the topology on Λ is the coordinatewise convergence topology, it follows that $\alpha_i^{\delta} \xrightarrow{\delta} \alpha_i$, for all $i \geq 1$, in $\mathbb{K}$. Also the system $\mathcal{G}$ which generates $\mathcal{F}$, can be chosen such that it contains normal hulls $\hat{S}$ of all its members and each member of $\mathcal{G}$ is $\sigma(\Lambda^{\times}(Y), \Lambda(X))$ relatively compact, as $\mathcal{F}$ is solid and compatible with the dual pair $\langle\Lambda(X), \Lambda^{\times}(Y)\rangle$. Thus, if we choose $S \in \mathcal{G}$, then $\hat{S} \in \mathcal{G}$ is $\sigma(\Lambda^{\times}(Y), \Lambda(X))$-relatively compact and so applying remark following Proposition 3.2, for given $\epsilon > 0$ and $\bar{x} \in \Lambda(X)$, there exists a positive integer N such that

$$\sum_{i \geq N+1} |\langle x_i, y_i\rangle| < \epsilon/6, \text{ for } \bar{y} \in S.$$

Now for $\bar{y} \in S$, we have

$$(*) \qquad \sum_{i \geq 1} |\langle \alpha_i^{\delta} x_i^{\delta} - \alpha_i x_i, y_i\rangle| \leq \sum_{i \geq 1} |\alpha_i^{\delta}|\, |\langle x_i^{\delta} - x_i, y_i\rangle| +$$

$$\sum_{i \geq 1} |\alpha_i^{\delta} - \alpha_i|\, |\langle x_i, y_i\rangle| \leq \sum_{i \geq 1} |\langle x_i^{\delta} - x_i, y_i\rangle| +$$

$$\sum_{i=1}^{N} |\alpha_i^{\delta} - \alpha_i|\, |\langle x_i, y_i\rangle| + 2 \sum_{i \geq N+1} |\langle x_i, y_i\rangle|$$

Since S, being $\sigma(\Lambda^{\times}(Y), \Lambda(X))$ bounded is $\eta(\Lambda^{\times}(Y), \Lambda(X))$-bounded by Proposition 3.3, Chapter 2, there exists $M > 0$ such that

$$\sum_{i \geq 1} |\langle x_i, y_i\rangle| < M, \text{ for } \bar{y} \in S.$$

Also, the convergence of $\bar{x}^{\delta}$ to $\bar{x}$ in $\Lambda(X)$ and of α_i's to α_i in $\mathbb{K}$, for each $i \geq 1$, guarantees the existence of a $\delta_o \in \Delta$ such that for $\delta \geq \delta_o$

$$\sum_{i \geq 1} |\langle x_i^{\delta} - x_i, y_i \rangle| < \epsilon/3$$

and

$$|\alpha_i^{\delta} - \alpha_i| < \epsilon/3M, \quad 1 \leq i \leq N .$$

Hence from (*) we get

$$\sum_{i \geq 1} |\langle \alpha_i^{\delta} x_i^{\delta} - \alpha_i x_i, y_i \rangle| < \epsilon, \text{ for } \delta \geq \delta_o$$

$$\Longrightarrow \bar{\alpha}^{\delta}\, \bar{x}^{\delta} \to \bar{\alpha}\, \bar{x}, \text{ or, } f((\bar{\alpha}^{\delta}, \bar{x}^{\delta})) \to f((\bar{\alpha}, \bar{x}))$$

$$\Longrightarrow f \text{ is continuous.}$$

Hence $f(A \times K)$ is a compact subset of $\Lambda(X)$. But $\hat{K} \subset f(A \times K)$ and so $\hat{K}$ is relatively compact. This completes the proof.

CHAPTER 6

MATRIX TRANSFORMATIONS ON GENERALIZED SEQUENCE SPACES

1. INTRODUCTION

This chapter mainly deals with the problem of representing an arbitrary continuous linear map on VVSS in terms of an infinite matrix of linear operators on the underlying spaces. We start this study in section 2, wherein we introduce the notion of a matrix transformation on VVSS in a manner different from Gregory [38], and obtain a few results concerning such maps and diagonal operators resulting from a sequence of linear operators on the underlying spaces. The study of diagonal operators is further carried over to the next section where we characterize their nuclearity and precompactness. In section 4 we introduce two kinds of simple character of VVSS, illustrate them with examples and study a few of their properties. The final section of this chapter incorporates results on matrix transformations involving nuclear and simple VVSS.

2. MATRIX TRANSFORMATIONS

Let us assume throughout this section that (X,T_X) and (Y,T_Y) are two Hausdorff locally convex spaces with corresponding duals X^* and Y^* respectively and $\langle\Lambda(X),\Lambda^{\times}(X^*)\rangle$

$\langle \Lambda(Y), \Lambda^{\times}(Y^*)\rangle$ form dual pairs of VVSS. For our notational convenience, we use the symbols $P_{i,X}$ and $R_{i,X}$, respectively for the projections from $\Lambda(X)$ to X and injections from X to $\Lambda(X)$ defined in section 2 Chapter 4.

We introduce

DEFINITIONS 2.1 : A linear map $Z: \Lambda(X) \to \Lambda(Y)$ is said to be a T_Y-<u>matrix transformation</u> or <u>matrix transformation from $\Lambda(X)$ to $\Lambda(Y)$ relative to T_Y</u> if there is a matrix $[Z_{ij}]$ of linear maps $Z_{ij}: X \to Y$, $i,j=1,2,\ldots$, such that for each $\bar{x} = \{x_i\}$ in $\Lambda(X)$, the series $\sum_{j\geq 1} Z_{ij}(x_j)$ converges to some element, say y_i, in Y with respect to T_Y, for each $i \geq 1$ and $y_i = \sum_{j\geq 1} Z_{ij}(x_j) = P_{i,Y}(Z(\bar{x}))$. For a matrix $Z \equiv [Z_{ij}]$ of linear maps $Z_{ij}: X \to Y$, $i,j \geq 1$, we define its <u>transpose</u> $Z^{\perp}$ as the transpose of the matrix of adjoint maps of Z_{ij}, i.e., $Z^{\perp} = [Z^*_{ji}]$.

Note : As mentioned in section 3 of Chapter 2, Gregory defines the notion of a matrix transformation, when Y is equipped with the weak topology $\sigma(Y,Y^*)$ and characterizes these in the form of Proposition 3.7, Chapter 2.

Concerning the matrix representation of continuous linear maps on VVSS, we prove

THEOREM 2.2 : Let $(\Lambda(X), \mathfrak{J})$ and $(\Lambda(Y), \mathfrak{J}')$ be two monotone,

barrelled GC-and GAK-spaces. Then a linear map $Z:\Lambda(X) \to \Lambda(Y)$ is $\mathcal{F}$-$\mathcal{F}'$ continuous if and only if it can be represented by a matrix $[Z_{ij}]$ of T_X-T_Y continuous linear maps Z_{ij} from X to Y.

PROOF : Let Z be a $\mathcal{F}$-$\mathcal{F}'$ continuous linear map from $\Lambda(X)$ to $\Lambda(Y)$. For $i,j \geq 1$, define $Z_{ij}:X \to Y$ by

$$Z_{ij} = (P_{i,Y} \circ Z \circ R_{j,X})$$

Clearly, Z_{ij}'s are T_X-T_Y continuous linear maps from X to Y. Making use of the GAK-character of the space $\Lambda(X)$, we have for any $\bar{x} = \{x_i\} \in \Lambda(X)$,

$$\bar{x}^{(n)} \to \bar{x}\text{, as } n \to \infty \text{ relative to } \mathcal{F} \text{ in } \Lambda(X)$$

$$\Longrightarrow \quad Z(\bar{x}^{(n)}) \to Z(\bar{x}) \text{ as } n \to \infty\text{, in } \mathcal{F}'.$$

But $Z(\bar{x}^{(n)}) = \sum_{j=1}^{n} Z(\delta_j^{x_j}) = \sum_{j=1}^{n} Z \circ R_{j,X}(x_j)$ and the continuity of $P_{i,Y}$, for each $i \geq 1$, imply that

$$\sum_{j=1}^{n} (P_{i,Y} \circ Z \circ R_{j,X})(x_j) \to P_{i,Y}(Z(\bar{x})) \text{ as } n \to \infty,$$

relative to T_Y in Y. Thus

$$P_{i,Y}(Z(\bar{x})) = \sum_{j\geq 1} Z_{ij}(x_j)$$

and therefore $Z \equiv [Z_{ij}]$ is a T_Y-matrix transformation from $\Lambda(X)$ to $\Lambda(Y)$.

Conversely, let Z be represented by a matrix $[Z_{ij}]$ where $Z_{ij}:X \to Y$ are T_X-T_Y continuous linear maps. Then Z_{ij}'s are $\sigma(X,X^*)$-$\sigma(Y,Y^*)$ continuous by Proposition 2.18, Chapter 2. As the spaces $\Lambda(X)$ and $\Lambda(Y)$ satisfy the hypothesis of Corollary 4.8, Chapter 4 we have $\Lambda^\times(X^*) = [\Lambda(X)]^*$ and $\Lambda^\times(Y^*) = [\Lambda(Y)]^*$. Also from the barrelled character of the spaces $(\Lambda(X),\mathfrak{J})$ and $(\Lambda(Y),\mathfrak{J}')$,

$$\mathfrak{J} = \tau(\Lambda(X), \Lambda^\times(X^*)) ,$$
$$\mathfrak{J}' = \tau(\Lambda(Y), \Lambda^\times(Y^*)) ,$$

and the space $(\Lambda^\times(X^*), \sigma(\Lambda^\times(X^*), \Lambda(X)))$ is sequentially complete. Applying Propositions 2.19 and 3.7 of Chapter 2, we conclude the $\mathfrak{J}$-$\mathfrak{J}'$ continuity of Z.

Next, we have

THEOREM 2.3 : Let $\Lambda(X)$ be monotone and $(\Lambda^\times(X^*), \sigma(\Lambda^\times(X^*), \Lambda(X)))$ be sequentially complete. If Z is a $\sigma(Y,Y^*)$-matrix transformation from $\Lambda(X)$ to $\Lambda(Y)$, represented by the matrix $[Z_{ij}]$ of $\sigma(X,X^*)$-$\sigma(Y,Y^*)$ continuous linear maps $Z_{ij}:X \to Y$, then $Z^\perp$ is a $\sigma(X^*,X)$-matrix transformation from $\Lambda^\times(Y^*)$ to $\Lambda^\times(X^*)$.

PROOF : Since Z is $\sigma(\Lambda(X), \Lambda^\times(X^*))$- $\sigma(\Lambda(Y), \Lambda^\times(Y^*))$ continuous by Proposition 3.7, Chapter 2, its adjoint Z^* would map $\Lambda^\times(Y^*)$ into $\Lambda^\times(X^*)$. So, the result would follow if we show that $Z^\perp = Z^*$.

Let us consider $\bar{g} = \{g_i\} \in \Lambda^{\times}(Y^*)$ and fix it. For $\bar{x} = \{x_i\} \in \Lambda(X)$, write $\bar{y} = Z(\bar{x})$, where $y_i = \sum_{j\geq 1} Z_{ij}(x_j)$, the convergence of the series being considered in $\sigma(Y,Y^*)$.

Then

$$\langle \bar{x}, Z^*(\bar{g})\rangle = \langle \bar{y}, \bar{g}\rangle = \sum_{n\geq 1} \langle \sum_{j\geq 1} Z_{nj}(x_j), g_n\rangle$$

$$= \sum_{n\geq 1} \sum_{j\geq 1} \langle Z_{nj}(x_j), g_n\rangle .$$

Thus

$$(*) \qquad \langle \bar{x}, Z^*(g)\rangle = \sum_{n\geq 1} \sum_{j\geq 1} \langle x_j, Z^*_{nj}(g_n)\rangle .$$

Since the above equality is true for any $\bar{x} \in \Lambda(X)$, it follows that the series $\sum_{j\geq 1} \langle x_j, Z^*_{nj}(g_n)\rangle$ converges for each $n\geq 1$. Hence the sequence $\bar{f}^n = \{Z^*_{nj}(g_n)\}_{j\geq 1}$ is in $(\Lambda(X))^{\beta}$.

Write

$$\bar{F}^n = \sum_{i=1}^{n} \bar{f}^i .$$

Then from the equality

$$|\langle \bar{x}, \bar{F}^n - \bar{F}^m\rangle| = |\sum_{i=m+1}^{n} \langle y_i, g_i\rangle|$$

it follows that $\{\bar{F}^n\}$ is a $\sigma((\Lambda(X))^{\beta}, \Lambda(X))$-Cauchy sequence and so there exists $\bar{f} \in (\Lambda(X))^{\beta}$, $\bar{f} \equiv \{f_i\}$, such that

$$\bar{F}^n \to \bar{f}, \text{ as } n \to \infty \text{ in } \sigma((\Lambda(X))^{\beta}, \Lambda(X)).$$

If $\bar{F}^n = \{F_{nj}\}_{j=1}^{\infty}$, then

$$F_{nj} = \sum_{i=1}^{n} Z^*_{ij}(g_n), \quad \forall\ j,n \geq 1.$$

Since $((\Lambda(X))^{\beta}, \sigma((\Lambda(X))^{\beta}, \Lambda(X)))$ is a GK-space when X^* is equipped with the topology $\sigma(X^*,X)$, we get

$$F_{nj} \to f_j, \text{ as } n \to \infty \text{ in } \sigma(X^*,X)$$

$$\Longrightarrow \qquad f_j = \sum_{i=1}^{\infty} Z^*_{ij}(g_i), \text{ in } \sigma(X^*,X)\text{-topology}$$

Thus $Z^{\perp}(\bar{g}) = \bar{f}$, where $\bar{f} \in (\Lambda(X))^{\beta} = \Lambda^{\times}(X^*)$, by monotonicity of $\Lambda(X)$. Also from (*), we have

$$\langle \bar{x}, Z^*(\bar{g})\rangle = \lim_{n\to\infty} \langle \bar{x}, \bar{F}^n \rangle = \langle \bar{x}, \bar{f}\rangle$$

$$\Longrightarrow \langle \bar{x}, Z^*(\bar{g}) \rangle = \langle \bar{x}, Z^{\perp}(\bar{g})\rangle.$$

This is true for all $\bar{x}$ in $\Lambda(X)$ and hence

$$Z^*(\bar{g}) = Z^{\perp}(\bar{g}), \text{ for } \bar{g} \in \Lambda^{\times}(Y^*).$$

Consequently, $Z^* = Z^{\perp}$ and therefore Z is a well defined map from $\Lambda^{\times}(Y^*)$ into $\Lambda^{\times}(X^*)$. Thus the proof is complete.

Note : Theorem 2.3 includes the result of Allen ([57], Proposition 3.3) for scalar case.

The above result leads to its converse in the form of

PROPOSITION 2.4 : Let $\Lambda(Y)$ be a perfect VVSS such that $(\Lambda(Y), \sigma(\Lambda(Y), \Lambda^{\times}(Y^*)))$ is sequentially complete. If Z is

a matrix of $\sigma(X,X^*)$- $\sigma(Y,Y^*)$ continuous linear maps $Z_{ij}:X \to Y$ such that its transpose $Z^{\perp}$ transforms $\Lambda^{\times}(Y^*)$ into $\Lambda^{\times}(X^*)$ relative to the topology $\sigma(X^*,X)$, then Z is a $\sigma(Y,Y^*)$-matrix transformation from $\Lambda(X)$ to $\Lambda(Y)$.

PROOF : Since $Z^{\perp\perp} = Z$, the result is immediate from Theorem 2.3.

Diagonal Transformations :

These transformations correspond to diagonal matrices of linear operators defined in the following

DEFINITION 2.5 : A linear map $Z: \Lambda(X) \to \Lambda(Y)$ is a <u>diagonal matrix transformation</u> or just <u>diagonal transformation</u> if it can be represented in the form of a matrix $[Z_{ij}]$ of linear maps Z_{ij} from X to Y such that $Z_{ij}=0$ for $i \neq j$, $Z_{ii} = Z_i$, $i \geq 1$; and for $\bar{x} = \{x_i\}$ in $\Lambda(X)$, $Z(\bar{x}) = \{Z_i(x_i)\}$.

We denote by Z^i and $Z^{(n)}$, where $i,n \in \mathbb{N}$, the maps from $\Lambda(X)$ to $\Lambda(Y)$ defined respectively by

$$Z^i(\bar{x}) = \{0,0,\ldots,Z_i(x_i),0,0,\ldots\} = \delta_i^{Z_i(x_i)}$$

and

$$Z^{(n)}(\bar{x}) = \{Z_1(x_1), Z_2(x_2),\ldots,Z_n(x_n),0,0,\ldots\}$$
$$= \sum_{i=1}^{n} Z^i(\bar{x})$$

or

$$Z^{(n)} = \sum_{i=1}^{n} Z^i .$$

We observe that a diagonal matrix transformation Z is associated with a sequence $\{Z_i\}$ of operators on underlying spaces. However, the problem of getting a diagonal matrix transformation on VVSS defined with the help of a given normal SVSS λ, from a given sequence $\{Z_i\}$ of operators on underlying spaces, is solved in

PROPOSITION 2.6 : Let $\{Z_n\}$ be an equicontinuous family of linear maps from X to Y. Then the map Z, defined by $Z(\bar{x}) = \{Z_n(x_n)\}$, is a continuous diagonal transformation from $\lambda(X)$ into $\lambda(Y)$.

PROOF : Since $\{Z_n\}$ is equicontinuous, for $q \in D_Y$, there exists $p \in D_X$ such that

$$q(Z_n(x)) \leq p(x), \quad \forall\, x \in X \text{ and } n \geq 1.$$

Thus, for $\{x_n\} \in \lambda(X)$, $\{q(Z_n(x_n))\} \in \lambda$, $\forall q \in D_Y$.
Hence $\{Z_n(x_n)\} \in \lambda(Y)$. Since the topology on λ is solid, the semi-norms $\{p_S : S \in \mathfrak{S}\}$ generating the topology on λ are monotone and so for a semi-norm $p_S \circ q \in D_{\lambda(Y)}$, we have

$$\begin{aligned}(p_S \circ q)(\{Z_n(x_n)\}) &= p_S\{q(Z_n(x_n))\} \\ &\leq p_S\{p(x_n)\} = (p_S \circ p)(\{x_n\})\end{aligned}$$

for $\bar{x} = \{x_n\} \in \lambda(X)$. Hence the continuity of $Z \equiv \{Z_n\}$, is established.

PROPOSITION 2.7 : In addition to the hypothesis of Proposition 2.6, assume that X and Y are Banach spaces. Then the adjoint Z^* of Z is the linear map $Z^{\perp}: \lambda^{\times}(Y^*) \to \lambda^{\times}(X^*)$, that is, $Z^* = Z^{\perp} \equiv \{Z_n^*\}$, where Z_n^*'s are the adjoint maps of Z_n's, $n \geq 1$.

PROOF : From Proposition 3.10, Chapter 2, $(\lambda(X))^* = \lambda^{\times}(X^*)$ and $(\lambda(Y))^* = \lambda^{\times}(X^*)$, and therefore, Z^* transforms $\lambda^{\times}(Y^*)$ into $\lambda^{\times}(X^*)$. Now, for $\bar{x} = \{x_n\} \in \lambda(X)$ and $\bar{g} = \{g_i\} \in \lambda^{\times}(Y^*)$, we have

$$
\begin{aligned}
\langle \bar{x}, Z^*(\bar{g}) \rangle &= \langle Z(\bar{x}), \bar{g} \rangle \\
&= \sum_{n \geq 1} \langle Z_n(x_n), g_n \rangle \\
&= \sum_{n \geq 1} \langle x_n, Z_n^*(g_n) \rangle \\
&= \langle \bar{x}, \{Z_n^*(g_n)\} \rangle .
\end{aligned}
$$

This is true for every $\bar{x} \in \lambda(X)$ and so $Z^* \equiv \{Z_n^*\} = Z^{\perp}$ and the proof is complete.

Remark :Proposition 2.7 also follows from Theorem 2.3.

PROPOSITION 2.8 : Let λ, equipped with $\|\ \|_{\lambda}$, be assumed to be a Banach space such that $\|e^n\|_{\lambda} = 1$, for $n \geq 1$, in addition to the hypothesis of Proposition 2.7. Then $\|Z\| = \sup_{n \geq 1} \|Z_n\|$, where the norms of Z and Z_n are the usual operator norm.

PROOF : Since Z_n's are equicontinuous, $\sup_{n\geq 1} ||Z_n|| < \infty$ and

$$||Z_n(x)||_Y \leq (\sup_{n\geq 1} ||Z_n||)||x||, \ \forall x \in X, \ n \geq 1.$$

For $\bar{x} = \{x_n\} \in \lambda(X)$, consider $||Z(\bar{x})||_{\lambda(Y)} = ||\{||Z_n(x_n)||_Y\}||_\lambda$.

As

$$||Z_n(x_n)||_Y \leq ||Z_n|| \ ||x_n|| \leq (\sup_{n\geq 1} ||Z_n||)||x_n||, \ \forall \ n\geq 1$$

and $|| \ ||_\lambda$ is monotone, we have

$$\begin{aligned} ||Z(\bar{x})||_{\lambda(Y)} &\leq ||\{(\sup_{n\geq 1} ||Z_n||)||x_n||\}||_\lambda \\ &= (\sup_{n\geq 1} ||Z_n||)||\{||x_n||\}||_\lambda \\ &= (\sup_{n\geq 1} ||Z_n||)||\bar{x}||_{\lambda(X)} . \end{aligned}$$

Thus

$$||Z|| \leq \sup_{n\geq 1} ||Z_n||$$

On the otherhand, if we choose $x \in X$ with $||x|| \leq 1$, then

$$\begin{aligned} ||\delta_n^x||_{\lambda(X)} &= ||\{0,0,\ldots,||x||,0,\ldots\}||_\lambda \\ &= ||x|| \ ||e^n||_\lambda . \end{aligned}$$

Since $||e^n||_\lambda = 1$, $||\delta_n^x|| \leq 1$, $\forall n \geq 1$. Therefore

$$\begin{aligned} ||Z|| &= \sup_{||\bar{x}||\leq 1} ||Z(\bar{x})||_{\lambda(Y)} \\ &\geq ||Z(\delta_n^x)||_{\lambda(Y)} \end{aligned}$$

for every $n \geq 1$ and each $x \in X$ with $||x|| \leq 1$. But

$$||Z(\delta_n^x)||_{\lambda(Y)} = ||Z_n(x)||_Y ||e^n||_\lambda = ||Z_n(x)||_Y, \ \forall\, n \geq 1.$$

Therefore,

$$||Z|| \geq ||Z_n||, \ \forall\, n \geq 1.$$

Hence, we conclude the result.

3. NUCLEAR AND PRECOMPACT DIAGONAL OPERATORS

In this section we consider the nuclearity and precompactness of diagonal maps in terms of component operators. We begin with

PROPOSITION 3.1: Let $\Lambda(X)$ and $\Lambda(Y)$ be respectively GC-normed and GK-normed spaces and $Z \equiv \{Z_i\}$, $Z_i: X \to Y$, be a diagonal map from $\Lambda(X)$ to $\Lambda(Y)$. If Z is nuclear, then each Z_i, $i \geq 1$, is nuclear.

PROOF : Follows from the continuity of $P_{i,Y}$, $R_{i,X}$, Proposition 4.3(i), Chapter 2 and the fact that $Z_i = P_{i,Y} \circ Z \circ R_{i,X}$, for each $i \geq 1$.

PROPOSITION 3.2 : Let $\Lambda(X)$ be a GK-normed space, $\Lambda(Y)$ a GC-normed space, and $\{Z_i\}$ a sequence of nuclear maps from X to Y. Then the map $Z^i: \Lambda(X) \to \Lambda(Y)$, $Z^i(\bar{x}) = \delta_i^{Z_i(x_i)}$, is nuclear for each $i \geq 1$ and hence the maps $Z^{(n)} = \sum_{i=1}^{n} Z^i$, for $n \geq 1$, are nuclear maps.

PROOF : Again the nuclearity of Z^i is clear from the relation $Z^i = R_{i,Y} \circ Z_i \circ P_{i,X}$, $i \geq 1$ and Proposition 4.3(i) of Chapter 2. Since $Z^{(n)} = \sum_{i=1}^{n} Z^i$, the nuclearity of $Z^{(n)}$ for each $n \geq 1$, follows.

Remark : It is clear from Propositions 3.1 and 3.2 that the nuclearity of the maps Z_i and Z^i, $i \geq 1$ are equivalent if both the spaces $\Lambda(X)$ and $\Lambda(Y)$ are GC-, GK-normed spaces.

We now concentrate upon the study of these diagonal maps on the VVSS $\lambda(X)$ which are defined in section 3 of Chapter 2. Since the nuclearity notion of an operator involves the topological dual of the space on which it is defined, the following result which is of independent interest, is useful to us for proving the results of this section.

LEMMA 3.3 : For a normed space $(X, \|\ \|_X)$, $(\lambda(X))^\times = \lambda^\times(X^*) = (\lambda(X))^*$.

PROOF : In view of Proposition 3.10 (ii), Chapter 2, we need prove the first equality. Clearly $\lambda^\times(X^*) \subset (\lambda(X))^\times$.

For the other inclusion, consider $\{f_i\} \in (\lambda(X))^\times$ and $\{\alpha_i\} \in \lambda$. Then we can find a sequence $\{x_i\} \subset X$ with $\|x_i\| \leq 1$, $i \geq 1$ such that

$$\|\alpha_i f_i\| \leq |\langle x_i, \alpha_i f_i \rangle| + \frac{1}{2^i}, \quad \forall\, i \geq 1.$$

Since λ is normal, $\{||\alpha_i x_i||\} \in \lambda$ and so $\sum_{i\geq 1} |\langle \alpha_i x_i, f_i \rangle|$ is finite. Hence $\sum_{i\geq 1} |\alpha_i| \; ||f_i|| < \infty$, and therefore $\{f_i\} \in \lambda^{\times}(X^*)$. This completes the proof.

PROPOSITION 3.4 : Let $(\lambda, || \; ||_\lambda)$ be an AK-, BK-normal sequence space and $(X, || \; ||_X)$, $(Y, || \; ||_Y)$ two Banach spaces. If $Z \equiv \{Z_i\}$ is a diagonal nuclear map from $\lambda(X)$ to $\lambda(Y)$, then each Z_i is nuclear and $\sum_{i\geq 1} N(Z_i) < \infty$.

PROOF : Let $Z: \lambda(X) \to \lambda(Y)$ and $Z \equiv \{Z_i\}$ be a diagonal nuclear map. Then there exist sequences $\{f^n\} \subset (\lambda(X))^*$ and $\{\bar{y}^{(n)}\} \subset \lambda(Y)$ with

$$\sum_{n\geq 1} ||f^n||_{(\lambda(X))^*} \, ||\bar{y}^n||_{\lambda(Y)} < \infty,$$

and

$$Z(\bar{x}) = \sum_{n\geq 1} \langle \bar{x}, f^n \rangle \, \bar{y}^n, \quad \forall \, \bar{x} \in \lambda(X).$$

In view of Proposition 4.7 Chapter 4 and Proposition 3.10, Chapter 2, we can identify each f^n with the sequence $\{f^n \circ R_{i,X}\} \in \lambda^{\times}(X^*)$ such that

$$Z(\bar{x}) = \sum_{n\geq 1} \langle \bar{x}, \{f^n \circ R_{i,X}\} \rangle \, \bar{y}^n$$

where

$$\sum_{n\geq 1} ||\{f^n \circ R_{i,X}\}||_{\lambda^{\times}(X^*)} \, ||\bar{y}^n||_{\lambda(Y)} < \infty.$$

Therefore, for $i\geq 1$,

$$Z^i(\bar{x}) = \sum_{n\geq 1} \langle x_i, f^n \circ R_{i,X}\rangle \, \bar{y}^n$$

and so

$$Z_i(x) = P_{i,Y}(Z^i(\delta_i^x)) = \sum_{n\geq 1} \langle x, f^n \circ R_{i,X}\rangle y_i^n, \quad x \in X, \ i \geq 1.$$

Once again applying Proposition 3.10, Chapter 2, we get for fixed $n \geq 1$,

$$\|\{f^n \circ R_{i,X}\}\|_{\lambda^\times(X^*)} \|\bar{y}^n\|_{\lambda(Y)}$$

$$= \Big(\sup_{\|\bar{\alpha}\|_\lambda \leq 1} \sum_{i\geq 1} |\alpha_i| \, \|f^n \circ R_{i,X}\|_{X^*}\Big) \|\{\|y_i^n\|_Y\}\|_\lambda \,.$$

Choose $\{\alpha_i\} \subset \mathbb{K}$ as follows

$$\alpha_i = \begin{cases} \dfrac{\|y_i^n\|_Y}{\|\{\|y_i^n\|_Y\}\|_\lambda} & \text{if } \|\{\|y_i^n\|_Y\}\|_\lambda \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\|\{f^n \circ R_{i,X}\}\|_{\lambda^\times(X^*)} \|\bar{y}^n\|_{\lambda(Y)} \geq \sum_{i\geq 1} \|y_i^n\|_Y \|f^n \circ R_{i,X}\|_{X^*} \,.$$

Hence

$$\begin{aligned} \sum_{n\geq 1} \|y_i^n\|_Y \|f^n \circ R_{i,X}\|_{X^*} &\leq \sum_{n\geq 1} \sum_{i\geq 1} \|y_i^n\|_Y \|f^n \circ R_{i,X}\|_{X^*} \\ &\leq \sum_{n\geq 1} \|\{f^n \circ R_{i,X}\}\|_{\lambda^\times(X^*)} \|\bar{y}^n\|_{\lambda(Y)} \\ &< \infty \,. \end{aligned}$$

Thus each Z_i is nuclear, for

$$N(Z_i) \leq \sum_{n\geq 1} ||y_i^n||_Y ||f^n \circ R_{i,X}||_{X^*} < \infty, \forall\, i \geq 1.$$

Moreover, this also leads to the convergence of the series $\sum_{i\geq 1} N(Z_i)$ as

$$\sum_{i\geq 1} N(Z_i) \leq \sum_{i\geq 1} \sum_{n\geq 1} ||y_i^n||_Y ||f^n \circ R_{i,X}||_{X^*} < \infty .$$

The proof is now complete.

Note : No doubt in Proposition 3.4, the nuclearity of each Z_i can also be shown with the help of Proposition 3.10 (i), Chapter 2 and Proposition 3.1; but for the convergence of $\sum_{i\geq 1} N(Z_i)$, one has to get involved with the nuclear representation of maps.

For concluding the nuclearity of Z from that of Z_i's, we have the following results:

PROPOSITION 3.5 : Let λ , X, Y be as in Proposition 3.4 and $Z = \{Z_i\}$ be a diagonal transformation from $\lambda(X)$ to $\lambda(Y)$ such that the maps $Z_i : X \to Y$, $i \geq 1$, are nuclear and $\sum_{i\geq 1} N(Z^i) < \infty$. Then the diagonal map Z is nuclear.

PROOF : From Proposition 3.10 (i), Chapter 2, the spaces $\lambda(X)$ and $\lambda(Y)$ are GK-, GAK- and GC-Banach spaces and so $Z^{(n)}$, $n \geq 1$ are nuclear maps (cf. Proposition 3.2) and

$$\lim_{n\to\infty} Z^{(n)}(\bar{x}) = Z(\bar{x}), \ \forall \ \bar{x} \in \ \lambda(X).$$

Also, for $m, n \in \mathbb{N}$ with $m > n$, the inequality

$$N(Z^{(m)} - Z^{(n)}) = N(\sum_{i=n+1}^{m} Z^i) \leq \sum_{i=n+1}^{m} N(Z^i),$$

and $\sum_{i \geq 1} N(Z^i) < \infty$, imply that $\{Z^{(n)}\}$ is a Cauchy sequence in the space of nuclear operators from $\lambda(X)$ to $\lambda(Y)$ with respect to the nuclear norm. Hence Z is nuclear by Proposition 4.3(ii), Chapter. Thus the result is established.

A variation of the above result is contained in

PROPOSITION 3.6 : Let λ, X, Y, $\lambda(X)$, $\lambda(Y)$ be as in the preceding proposition and $Z \equiv \{Z_i\}$ be a diagonal map from $\lambda(X)$ to $\lambda(Y)$ such that each Z_i is nuclear and $\sum_{i \geq 1} N(Z_i) < \infty$. Further, assume that $\{e^i\}$ is a bounded sequence in both the spaces λ and $\lambda^{\times}$. Then the map Z is nuclear.

PROOF : Since each Z_i is nuclear, we can find sequences $\{f_i^n\} \subset X^*$ and $\{y_i^n\} \subset Y$ such that for $i \geq 1$,

$$Z_i(x) = \sum_{n \geq 1} \langle x, f_i^n \rangle y_i^n, \quad \forall \ x \in X$$

and

$$\sum_{n \geq 1} ||f_i^n||_{X^*} \, ||y_i^n||_Y < \infty.$$

Therefore, for $i \geq 1$, we have

$$Z^i(\bar{x}) = \sum_{n\geq 1} \langle \bar{x}, \delta_i^{f_i^n} \rangle \, \delta^{y_i^n}, \; \bar{x} \in \lambda(X),$$

the convergence of the series being considered in the norm topology of $\lambda(Y)$. Clearly

$$\{\delta_i^{f_i^n}\} \subset \lambda^\times(X^*) = (\lambda(X))^*, \; \delta^{y_i^n} \subset \lambda(Y)$$

and

$$\|\delta_i^{f_i^n}\|_{\lambda^\times(X^*)} \|\delta_i^{y_i^n}\|_{\lambda(Y)} \leq KK' \|f_i^n\|_{X^*} \|y_i^n\|_Y, \; \forall \; n\geq 1, \; i\geq 1$$

where K and K' are the numbers such that $\|e^i\|_\lambda \leq K$ and $\|e^i\|_{\lambda^\times} \leq K'$. Thus, each Z^i is nuclear and

$$N(Z^i) \leq KK' N(Z_i), \; \forall i \geq 1.$$

Hence $\sum_{i\geq 1} N(Z^i) < \infty$. Now we proceed as in the preceding proposition to conclude the nuclearity of Z.

Concerning the precompact diagonal maps, we prove

PROPOSITION 3.7 : Let $(X, \|\ \|_X)$, $(Y, \|\ \|_Y)$ be two normed spaces and λ equipped with a monotone norm $\|\ \|_\lambda$, be a normed, K-, AK-, normed space such that $\|e^i\|_\lambda = 1$, for each $i \geq 1$. Then a diagonal map $Z = \{Z_i\}$ from $\lambda(X)$ into $\lambda(Y)$ is precompact if and only if each $Z_i : X \to Y$, is precompact and $\|Z_i\| \to 0$, as $i \to \infty$, where $\|\ \|$ denotes the operator norm.

PROOF : Let us denote by B and $B_{\lambda(X)}$ the unit balls in X and $\lambda(X)$ respectively.

Let us first assume the precompactness of Z. Then $Z(B_{\lambda(X)})$ is a precompact set in Y. In order to show the precompactness of Z_i, it is sufficient to prove $Z_i(B) \subset P_{i,Y}(Z(B_{\lambda(X)}))$, $\forall i \geq 1$. Therefore, consider a point y in $Z_i(B)$. Then

$$y = Z_i(x), \text{ for some } x \in B$$

and

$$\|\delta_i^x\|_{\lambda(X)} = \|\{0,0,\ldots,\|x\|,0,\ldots\}\|_\lambda$$
$$= \|x\| \, \|e^i\|_\lambda \leq 1.$$

Thus

$$y = P_{i,Y}(Z(\delta_i^x)) \in P_{i,Y}(Z(B_{\lambda(X)})) .$$

Consequently, $Z_i(B) \subset P_{i,Y}(Z(B_{\lambda(X)}))$ and each Z_i is precompact.

For showing $\|Z_i\| \to 0$ as $i \to \infty$, choose $\epsilon > 0$. Then there exists a sequence $\{x_i\} \subset B$ with

$$\|Z_i\| < \|Z_i(x_i)\| + \epsilon/2, \ \forall i \geq 1.$$

Since $\|\delta_i^{x_i}\|_{\lambda(X)} = \|x\| \leq 1$, $\forall\, i \geq 1$, and Z is precompact, it follows that the set $\{Z(\delta_i^{x_i}) : i \geq 1\}$ is precompact in $\lambda(Y)$. Hence we can find finite indices $n_1, n_2, \ldots, n_m$ such that for $i \geq 1$ there exists n_j, $1 \leq j \leq m$ such that

$$\|Z(\delta_i^{x_i}) - Z(\delta_{n_j}^{x_{n_j}})\|_{\lambda(Y)} < \epsilon/2.$$

Choose $n_o = \max(n_1, n_2, \ldots, n_m)$. Then, the monotonocity of $\|\ \|_\lambda$ along with the following equalities

$$\|Z_i(x_i)\|_Y = \|Z(\delta_i^{x_i})\|_{\lambda(Y)} = \|\{0,0,\ldots,\|Z_i(x_i)\|,0,\ldots\}\|_\lambda,$$

and

$$\|Z(\delta_i^{x_i})-Z(\delta_{n_j}^{x_{n_j}})\|_{\lambda(Y)} = \|\{0,0,\ldots,\|Z_{n_j}(x_{n_j})\|,0,0,\ldots$$
$$\|Z_i(x_i)\|,0,0,\ldots\}\|_\lambda, \text{ for } i > n_o$$

implies

$$\|Z_i(x_i)\|_Y \leq \|Z(\delta_i^{x_i})-Z(\delta_{n_j}^{x_{n_j}})\|_{\lambda(Y)} < \epsilon/2, \text{ for } i > n_o.$$

Hence

$$\|Z_i\| < \epsilon, \text{ for } i > n_o.$$

This completes the proof of the necessity part.

For converse, let us assume that the operators Z_i's are precompact. Thus for $i \geq 1$, each $Z_i(B)$ is a precompact set in Y. Consequently, the sets $(R_{i,Y} \circ Z_i)(B)$ are precompact in $\lambda(Y)$ for each $i \geq 1$. Also, for $\bar{x} \in \lambda(X)$, the inequality

$$\|\delta_i^{x_i}\|_{\lambda(X)} \leq \|\bar{x}\|_{\lambda(X)}, \quad \forall\, i \geq 1$$

and $\|x_i\|_X = \|\delta_i^{x_i}\|_{\lambda(X)}$, imply that for $\bar{x} \in B_{\lambda(X)}$, $x_i \in B$, $\forall\, i \geq 1$. Thus $Z^i(B_{\lambda(X)}) \subset (R_{i,Y} \circ Z_i)(B)$, $\forall\, i \geq 1$ and therefore each Z^i is precompact. Since $Z^{(n)} = \sum_{i=1}^{n} Z^i$, we conclude the precompactness of $Z^{(n)}$ for all $n \geq 1$. Now consider

$$||Z-Z^{(n)}|| = \sup_{\bar{x}\in B_{\lambda(X)}} ||\{0,0,\ldots,||Z_{n+1}(x_{n+1})||_Y ,$$

$$||Z_{n+2}(x_{n+2})||_Y,\ldots\}||_\lambda$$

$$\leq (\sup_{i>n+1} ||Z_i||) \sup_{\bar{x}\in B_{\lambda(X)}} ||\{0,0,\ldots,||x_{n+1}||_X ,$$

$$||x_{n+2}||_X,\ldots\}||_\lambda$$

$$\leq \sup_{i>n+1} ||Z_i|| .$$

As $||Z_i|| \to 0$ when $i \to \infty$, it follows that $\sup_{i>n+1} ||Z_i|| \to 0$ as $n \to \infty$. Thus $Z^{(n)} \to Z$ as $n \to \infty$ in the operator norm and so Z is precompact by Proposition 4.2, Chapter 2

4. SIMPLE GENERALIZED SEQUENCE SPACES

This section is devoted to the study of simple VVSS introduced in

DEFINITION 4.1 : Let (X,T) be an l.c. TVS and $\Lambda(X)$ a VVSS equipped with a locally convex topology $\mathfrak{F}$. Then $(\Lambda(X),\mathfrak{F})$ is said to be (i) <u>simple</u> if for each $\mathfrak{F}$-bounded subset A of $\Lambda(X)$, there exists an element $\bar{x}$ in $\Lambda(X)$ such that A is contained in the normal hull of the set $\{\bar{x}\}$, that is, $A \subset \widehat{\{\bar{x}\}}$; and (ii) <u>s-simple</u> if for $\mathfrak{F}$-bounded $A \subset \Lambda(X)$, there is an element $\bar{x} = \{x_i\} \in \Lambda(X)$ such that for each $p \in D_X$ and $\bar{y} = \{y_i\}$ in A, $p(y_i) \leq p(x_i)$, for each $i \geq 1$.

Note : It is clear from the above definition that every simple VVSS is s-simple. These two notions are the same in the case of SVSS, that is, when we restrict X to be equal to $\mathbb{K}$. However, it is not true in general as we will see in the sequel.

Let us now illustrate a simple VVSS in

EXAMPLE 4.2 : Consider the VVSS $\ell^{\infty\ \infty}$ defined as

$$\ell^{\infty\ \infty} = \{ [x_{ij}] : x_{ij} \in \mathbb{K},\ i,j \geq 1 \text{ and }$$

$$\sup \{ |x_{ij}| : i,j \geq 1 \} < \infty \}$$

and equip it with the norm

$$|| [x_{ij}] || = \sup \{ |x_{ij}| : i,j \geq 1 \} .$$

Let A be a bounded subset of $\ell^{\infty}(\ell^{\infty})$. Then there exists $M > 0$ such that $|x_{ij}| \leq M$, for all $i,j \geq 1$ and all $[x_{ij}]$ in A. Define $[b_{ij}] \in \ell^{\infty}(\ell^{\infty})$ by

$$b_{ij} = M, \ \forall\ i,j \geq 1.$$

Clearly A is contained in the normal hull of $[b_{ij}]$. Thus $\ell^{\infty\ \infty}$ is a simple normed VVSS.

For s-simple VVSS, we have the following general result:

PROPOSITION 4.3 : Let λ be a normal simple SVSS equipped with a solid topology and $(X, \| \ \|)$ be a normed space. Then $\lambda(X)$ is s-simple.

PROOF : Let us consider a bounded subset A of $\lambda(X)$. Define $P: \lambda(X) \to \lambda$ by

$$P(\{x_i\}) = \{\|x_i\|\}, \ \forall \ \{x_i\} \in \lambda(X)$$

Then P is continuous by Proposition 3.10 (iii), Chapter 2 and hence the set $P(A)$ is bounded in λ. Since λ is simple, there exists $\{\alpha_i\} \in \lambda$ such that

$$\|x_i\| \leq |\alpha_i|, \ \forall \ i \geq 1 \quad \text{and} \quad \{x_i\} \text{ in } A.$$

Choose $y_i \in X$ such that $\|y_i\| = |\alpha_i|$, $i \geq 1$. Thus we have $\{y_i\} \in \lambda(X)$ such that

$$\|x_i\| \leq \|y_i\|, \ \forall \ i \geq 1 \quad \text{and} \quad \{x_i\} \in A.$$

Hence $\lambda(X)$ is s-simple.

The above proposition is used in the following example which contains an s-simple space that is not simple.

EXAMPLE 4.4 : For a normed linear space $(X, \| \ \|)$ with $\dim(X) \geq 2$, consider the VVSS $\ell^\infty(X)$ as defined in Chapter 3. Let $\ell^\infty(X)$ be equipped with the norm $\| \ \|_\infty$, where

$$\|\{x_i\}\|_\infty = \sup_{i \geq 1} \|x_i\|, \ \forall \ \{x_i\} \in \ell^\infty(X).$$

Then the space $\ell^{\infty}(X)$ is s-simple by Proposition 4.3; but it is not simple. Indeed, for any two linearly independent elements x and y in X, consider the set A containing sequences $\bar{x}$ such that

$$\bar{x} \in A \iff \bar{x} = \left\{\frac{x}{(2n)^i}\right\}_i \quad \text{or} \quad \bar{x} = \left\{\frac{y}{(2n+1)^i}\right\}_i$$

for $n=1,2,3\ldots$. Clearly, A is a bounded subset of $\ell^{\infty}(X)$ and is not contained in the normal hull of any point of the space $\ell^{\infty}(X)$. For if it does, then there is an element $\bar{z} = \{z_i\} \in \ell^{\infty}(X)$ such that $A \subset \widehat{\{\bar{z}\}}$ and therefore for the sequences $\{\frac{x}{2}, \frac{x}{2^2}, \frac{x}{2^3}, \ldots\}$ and $\{\frac{y}{3}, \frac{y}{3^2}, \frac{y}{3^3}, \ldots\}$ in A we have $\{\alpha_i\}$ and $\{\beta_i\} \subset \mathbb{K}$ with $|\alpha_i| \leq 1$, $|\beta_i| \leq 1$ $\forall\, i \geq 1$ such that $\frac{x}{2} = \alpha_1 z_1$ and $\frac{y}{3} = \beta_1 z_1$. This contradicts the linear independence of x and y.

The rest of this section is now devoted to investigating several properties of simple generalized sequence spaces which we consider corresponding to a dual pair $\langle X,Y\rangle$ of vector spaces X and Y. We start with

PROPOSITION 4.5 : Let $\langle \Lambda(X), \mu(Y)\rangle$ be a dual system, where $\mu(Y)$ is a normal subspace of $\Lambda^{\times}(Y)$. Assume that $(\Lambda(X), \sigma(\Lambda(X),\mu(Y)))$ is simple and $\{\overset{n}{\bar{x}}\}$ a sequence and $\bar{x}$ a point in $\Lambda(X)$. Then (i) $\{\overset{n}{\bar{x}}\}$ converges to $\bar{x}$ in $\sigma(\Lambda(X),\mu(Y))$ if and only if $\{\overset{n}{\bar{x}}\}$ is bounded and the sequence $\{x_i^n\}$ converges

to x_i in $\sigma(X,Y)$, for each $i \geq 1$. (ii) $\{\bar{x}^n\}$ is $\sigma(\Lambda(X), \mu(Y))$-Cauchy if and only if $\{\bar{x}^n\}$ is bounded and $\{x_i^n\}$ is $\sigma(X,Y)$-Cauchy in X, for each $i \geq 1$.

PROOF : (i) Since the necessity part is clear, we need prove the sufficiency. Therefore assume that $\{\bar{x}^n\}$ is bounded in $\Lambda(X)$. Then there exists $\bar{u} \in \Lambda(X)$ such that $\{\bar{x}^n\} \subset \widehat{\{\bar{u}\}}$. Thus we can find scalars α_i^n, with $|\alpha_i^n| \leq 1$, $i,n \geq 1$ such that

$$x_i^n = \alpha_i^n u_i, \ \forall \ n,i \geq 1.$$

Let us consider an element $\bar{y} \in \mu(Y)$ and choose $\epsilon > 0$. Then we can find a positive integer N such that

$$\sum_{i>N} |\langle u_i, y_i\rangle| < \epsilon/4$$

and

$$\sum_{i>N} |\langle x_i, y_i\rangle| < \epsilon/4.$$

Also, from the $\sigma(X,Y)$-convergence of x_i^n to x_i, as $n \to \infty$, $\forall \ i \geq 1$, there exists a positive integer M such that

$$\sum_{i=1}^{N} |\langle x_i^n - x_i, y_i\rangle| < \epsilon/2, \ \forall \ n \geq M.$$

Hence, for $n \geq M$

$$\begin{aligned} q_{\bar{y}}(\bar{x}^n - \bar{x}) &\leq \sum_{i=1}^{N} |\langle x_i^n - x_i, y_i\rangle| + \sum_{i>N} |\langle x_i^n, y_i\rangle| + \sum_{i>N} |\langle x_i, y_i\rangle| \\ &\leq \sum_{i=1}^{N} |\langle x_i^n - x_i, y_i\rangle| + \sum_{i>N} |\langle u_i, y_i\rangle| + \sum_{i>N} |\langle x_i, y_i\rangle| \\ &< \epsilon . \end{aligned}$$

Thus $\bar{x}^n \to \bar{x}$, as $n \to \infty$ in $\sigma(\Lambda(X),\mu(Y))$ and (i) follows.

The proof of (ii) is similar to (i) and so it is omitted.

Next we have

PROPOSITION 4.6 : For the dual pair $\langle\Lambda(X), \Lambda^{\times}(Y)\rangle$ of VVSS, let $(\Lambda^{\times}(Y), \sigma(\Lambda^{\times}(Y), \Lambda(X)))$ be simple. Then every $\sigma(\Lambda(X), \Lambda^{\times}(Y))$-convergent (resp. Cauchy) sequence in $\Lambda(X)$ is $\beta(\Lambda(X), \Lambda^{\times}(Y))$-convergent (resp. Cauchy).

PROOF : Once again we prove the result for convergent sequences and the result for Cauchy sequences would follow on similar lines. Let $\{\bar{x}^n\}$ be a sequence in $\Lambda(X)$ such that $\bar{x}^n \to 0$, as $n \to \infty$ in $\sigma(\Lambda(X), \Lambda^{\times}(Y))$. Then $\bar{x}^n \to 0$, as $n \to \infty$ in $\eta(\Lambda(X), \Lambda^{\times}(Y))$ by Proposition 3.5 Chapter 2. Therefore for an arbitrary $\bar{y}$ in $\Lambda^{\times}(Y)$ and $\epsilon > 0$, there exists a positive integer N such that

$$\sum_{i\geq 1} |\langle x_i^n, y_i\rangle| < \epsilon, \ \forall\, n \geq N.$$

Now if B is any $\sigma(\Lambda^{\times}(Y), \Lambda(X))$-bounded subset of $\Lambda^{\times}(Y)$, there exists a $\bar{y} \in \Lambda^{\times}(Y)$ such that $B \subset \widehat{\{\bar{y}\}}$. For $n \geq N$, consider

$$p_B(\bar{x}^n) = \sup_{\bar{z}\in B} \Big|\sum_{i\geq 1} \langle x_i^n, z_i\rangle\Big|$$

$$\leq \sum_{i\geq 1} |\langle x_i^n, y_i\rangle|$$

$$< \epsilon.$$

Hence $\bar{x}^n \to 0$, as $n \to \infty$ in $\beta(\Lambda(X), \Lambda^\times(Y))$ and the proof is complete.

The following result which will be useful in our later result, is of independent interest and generalizes a corresponding result of Cać ([90], Proposition 3).

PROPOSITION 4.7 : Let $\langle\Lambda(X), \mu(Y)\rangle$ form a dual pair, where $\mu(Y)$ is a normal subspace of $\Lambda^\times(Y)$. Then $n(\Lambda(X), \mu(Y))$ is compatible with the dual pair $\langle\Lambda(X),\mu(Y)\rangle$.

PROOF : In order to prove the result, it is sufficient to show that the normal hulls of singleton sets in $\mu(Y)$ are $\sigma(\mu(Y), \Lambda(X))$-compact. Let us, therefore, consider a point $\bar{y} \in \mu(Y)$ and denote by M its normal hull. Let $\{\bar{x}^\delta : \delta \in \Delta\}$ be a net in M. Then we can find scalars α_i^δ with $|\alpha_i^\delta| \leq 1$, for $i \geq 1$, $\delta \in \Delta$ such that $x_i^\delta = \alpha_i^\delta y_i$, $i \geq 1$, $\delta \in \Delta$.

If B_∞ stands for the closed unit ball of ℓ^∞ then $\bar{\alpha}^\delta = \{\alpha_i^\delta\}$, is a net in B_∞ which is $\sigma(\ell^\infty, \ell^1)$-compact by Proposition 2.6, Chapter 2. Also, for each $i \geq 1$, $\{\alpha_i^\delta : \delta \in \Delta\}$ is a net in the compact unit ball u of scalars. Therefore, we can find a sequence $\{\alpha_i\} \subset u$ and a sequence $\{\bar{\alpha}^n\}$ from the net $\{\bar{\alpha}^\delta : \delta \in \Delta\}$ such that $\alpha_i^n \to \alpha_i$, as $n \to \infty$, for each $i \geq 1$. Thus making use of Theorem 2.2, Chapter 5 for scalar case (cf. also [70], p. 415) we derive the convergence of $\{\bar{\alpha}^n\}$ to $\bar{\alpha} = \{\alpha_i\}$ in the topology $\sigma(\ell^\infty, \ell^1)$.

Let us now consider a point $\bar{x} = \{x_i\}$ in $\Lambda(X)$. Then $\{\langle x_i, y_i\rangle\} \in \ell^1$ and so for $\epsilon > 0$, we can find a natural number n_0 such that

$$\left| \sum_{i \geq 1} (\alpha_i^n - \alpha_i) \langle x_i, y_i\rangle \right| < \epsilon, \quad n \geq n_0$$

or,

$$\left| \sum_{i \geq 1} \langle x_i, (\alpha_i^n - \alpha_i) y_i\rangle \right| < \epsilon, \quad n \geq n_0 .$$

Thus $\overset{n}{\bar{\alpha}}\, \bar{y} = \{\alpha_i^n y_i\}$ tends to $\bar{\alpha}\, \bar{y} = \{\alpha_i y_i\}$ as $n \to \infty$ in $\sigma(\mu(Y), \Lambda(X))$. Clearly $\bar{\alpha}\, \bar{y} \in M$ and it is an adherent point of the net $\{\bar{x}^{\delta} : \delta \in \Delta\}$. Hence M is $\sigma(\mu(Y), \Lambda(X))$-compact. This completes the proof.

PROPOSITION 4.8 : If $(\Lambda(X), \sigma(\Lambda(X), \Lambda^{\times}(Y)))$ is a simple sequence space, then so is $(\Lambda^{\times\times}(X), \sigma(\Lambda^{\times\times}(X), \Lambda^{\times}(Y)))$.

PROOF : Let A be a $\sigma(\Lambda^{\times\times}(X), \Lambda^{\times}(Y))$-bounded subset of $\Lambda^{\times\times}(X)$. Then, by Proposition 4.7, A is $\eta(\Lambda^{\times\times}(X), \Lambda^{\times}(Y))$-bounded and so for $\bar{y} \in \Lambda^{\times}(Y)$, there exists a positive constant K depending on $\bar{y}$ such that

$$(*) \qquad \sum_{i \geq 1} |\langle x_i, y_i\rangle| \leq K, \ \forall\, \bar{x} \in A.$$

Define $E = \{\bar{x}^{(n)} : n \geq 1,\ \bar{x} \in A\}$, that is, E is the set of sections of members of A. Clearly $E \subset \Lambda(X)$ and is $\sigma(\Lambda(X), \Lambda^{\times}(Y))$-bounded by $(*)$. Thus there exists $\bar{u} \in \Lambda(X)$ such that $E \subset \widehat{\{\bar{u}\}}$, which in turn implies that $A \subset \widehat{\{\bar{u}\}}$. Hence $(\Lambda^{\times\times}(X), \sigma(\Lambda^{\times\times}(X), \Lambda^{\times}(Y)))$ is simple.

From the proof of the above result, one immediately derives the following

PROPOSITION 4.9 : If $(\Lambda(X), \sigma(\Lambda(X), \Lambda^{\times}(Y)))$ is a normal simple sequence space, then $\Lambda(X)$ is perfect.

PROOF : In the proof of the preceding proposition, consider A to be a singleton set in $\Lambda^{\times\times}(X)$, that is, if $A = \{\bar{x}\}$, where $\bar{x} \in \Lambda^{\times\times}(X)$, we can find $\bar{u} \in \Lambda(X)$ and $\{\alpha_i\} \subset \mathbb{K}$ such that $\bar{x} = \{\alpha_i u_i\}$. Thus $\bar{x} \in \Lambda(X)$ as $\Lambda(X)$ is normal. Hence $\Lambda(X) = \Lambda^{\times\times}(X)$ and so it is perfect.

The above result leads to

PROPOSITION 4.10 : If $(X, \sigma(X,Y))$ is (sequentially) complete and $(\Lambda(X), \sigma(\Lambda(X), \Lambda^{\times}(Y)))$ is normal simple sequence space, then $(\Lambda(X), \eta(\Lambda(X), \Lambda^{\times}(Y)))$ is (sequentially) complete.

PROOF : The result is immediate from Proposition 4.9 and Proposition 2.1, Chapter 3.

The above proposition along with Proposition 3.5, Chapter 2 and Remark of Proposition 2.1, Chapter 3, yield

PROPOSITION 4.11 : Let $(X, \sigma(X,Y))$ be sequentially complete. If $(\Lambda(X), \sigma(\Lambda(X), \Lambda^{\times}(Y)))$ is a normal simple VVSS, then it is sequentially complete.

PROOF : Since $\sigma(\Lambda(X), \Lambda^{\times}(Y))$ and $\eta(\Lambda(X), \Lambda^{\times}(Y))$ convergent

sequences and Cauchy sequences are the same in $\Lambda(X)$ by Proposition 3.5, Chapter 2, the result follows from Proposition 4.10.

5. MATRIX TRANSFORMATIONS ON SIMPLE AND NUCLEAR SEQUENCE SPACES

This section is devoted to the study of matrix transformations which involve simple and nuclear VVSS. We assume throughout this section that (X, T_X) and (Y, T_Y) are two Hausdorff l.c.TVS with duals X^* and Y^* respectively. Then our first result in this direction is

THEOREM 5.1 : Let $\Lambda(X)$ and $\Lambda(Y)$ be normal VVSS. Assume that $(\Lambda^{\times}(X^*), \sigma(\Lambda^{\times}(X^*), \Lambda(X)))$ and $(Y, \sigma(Y, Y^*))$ are sequentially complete spaces and $(\Lambda(Y), \sigma(\Lambda(Y), \Lambda^{\times}(Y^*)))$ is simple. If $Z = [Z_{ij}]$ is an infinite matrix of $\sigma(X,X^*)$- $\sigma(Y,Y^*)$ continuous linear maps $Z_{ij}: X \to Y$, then the following statements are equivalent:

(i) Z is $\sigma(Y,Y^*)$-matrix transformation from $\Lambda(X)$ into $\Lambda(Y)$.

(ii) $Z^{\perp}$ is a $\sigma(X^*,X)$-matrix transformation from $\Lambda^{\times}(Y^*)$ into $\Lambda^{\times}(X^*)$.

(iii) For each $\{x_i\} \in \Lambda(X)$, there exists $\{y_i\} \in \Lambda(Y)$ such that

$$\sum_{j \geq 1} |\langle Z_{ij}(x_j), f\rangle| \leq |\langle y_i, f\rangle|,$$

for each $i \geq 1$ and $f \in Y^*$.

PROOF : (i) <==> (ii). Follows from Theorem 2.3 and Proposition 2.4.

(i) ==> (iii). For $\bar{x} \in \Lambda(X)$, define

$$K_{\bar{x}} = \{\bar{u} : \bar{u} = \{u_i\} \in \Omega(X),\ u_i = \alpha_i x_i,\ |\alpha_i| = 1,\ i \geq 1\} .$$

Clearly, $K_{\bar{x}}$ being $\eta(\Lambda(X), \Lambda^{\times}(X^*))$-bounded subset of $\Lambda(X)$, is $\sigma(\Lambda(X), \Lambda^{\times}(X^*))$-bounded by Proposition 4.7. Also from Proposition 3.7, Chapter 2, Z is $\sigma(\Lambda(X), \Lambda^{\times}(X^*))$- $\sigma(\Lambda(Y), \Lambda^{\times}(Y^*))$ continuous and, therefore, $Z(K_{\bar{x}})$ is $\sigma(\Lambda(Y), \Lambda^{\times}(Y^*))$-bounded in $\Lambda(Y)$ which is simple. Thus there exists an element $\bar{y} \in \Lambda(Y)$ with $Z(K_{\bar{x}}) \subset \{\hat{\bar{y}}\}$. Consequently, for $\bar{u} \in K_{\bar{x}}$, $Z(\bar{u}) = \{\alpha_i y_i\}$, for some $\{\alpha_i\} \subset \mathbb{K}$ with $|\alpha_i| \leq 1$, $i \geq 1$. Hence for $f \in Y^*$ and $i \geq 1$,

$$\left| \sum_{j \geq 1} \langle Z_{ij}(u_j), f\rangle \right| \leq |\langle y_i, f\rangle| .$$

Choose $\beta_{ij} \in \mathbb{K}$ with $|\beta_{ij}| = 1$ such that $|\langle Z_{ij}(u_j), f\rangle| = \langle Z_{ij}(u_j), f\rangle \beta_{ij}$ for $i,j \geq 1$. Then for each $i \geq 1$, the sequence $\{\beta_{ij} u_j\} \in K_{\bar{x}}$ and so

$$\sum_{j \geq 1} |\langle Z_{ij}(u_j), f\rangle| \leq |\langle y_i, f\rangle| .$$

This proves (iii).

(iii) ==> (i). Let $\bar{x} \in \Lambda(X)$. Define

$$s_n^i = \sum_{j=1}^{n} Z_{ij}(x_j),\ \forall\ i,n \geq 1.$$

Obviously, $s_n^i \in Y$ for each $i,n \geq 1$. Also, for $f \in Y^*$, the inequality

$$|\langle s_n^i - s_m^i, f\rangle| = \Big|\sum_{j=m+1}^{n} \langle Z_{ij}(x_j), f\rangle\Big|$$

$$\leq \sum_{j=m+1}^{n} |\langle Z_{ij}(x_j), f\rangle|$$

and the condition (iii) imply that $\{s_n^i : n \geq 1\}$ is a weak Cauchy sequence in Y, for each $i \geq 1$. Since Y is $\sigma(Y,Y^*)$ sequentially complete, there exist $z_i \in Y$, $i \geq 1$ such that for $f \in Y^*$ and $i \geq 1$,

$$\Big|\sum_{j\geq 1} \langle Z_{ij}(x_j), f\rangle\Big| = |\langle z_i, f\rangle|$$

Hence

$$(*) \qquad |\langle z_i, f\rangle| \leq |\langle y_i, f\rangle|, \ \forall\, f \in Y^* \text{ and } i \geq 1.$$

Thus for $\bar{f} \in \Lambda^\times(Y^*)$, it follows from $(*)$ that

$$\sum_{i\geq 1} |\langle z_i, f_i\rangle| \leq \sum_{i\geq 1} |\langle y_i, f_i\rangle| < \infty.$$

Consequently, $\bar{z} = \{z_i\} \in \Lambda^{\times\times}(Y)$. But $\Lambda(Y)$, being simple and normal, is perfect by Proposition 4.9 and so $\bar{z} \in \Lambda(Y)$. Clearly, $\bar{z} = Z(\bar{x})$. Hence (i) follows. The proof is now completely established.

The last result of this section is contained in

THEOREM 5.2 : Let Y be weakly sequentially complete and $\Lambda(X)$

be a normal VVSS with $(\Lambda^{\times}(X^*), \sigma(\Lambda^{\times}(X^*), \Lambda(X)))$ simple and sequentially complete. Assume that $(\mu(Y), \eta(\mu(Y),\mu^{\times}(Y^*)))$ is a nuclear perfect VVSS. If $Z \equiv [Z_{ij}]$ is a matrix of $\sigma(X,X^*)$-$\sigma(Y,Y^*)$ continuous linear maps $Z_{ij}: X \to Y$, then the following are equivalent:

(i) Z is a $\sigma(Y,Y^*)$-matrix transformation from $\Lambda(X)$ into $\Lambda(Y)$.

(ii) $Z^{\perp}$ is a $\sigma(X^*,X)$-matrix transformation from $\mu^{\times}(Y^*)$ into $\Lambda^{\times}(X^*)$.

(iii) For each $\bar{g} \in \mu^{\times}(Y^*)$, there exists $\bar{f} \in \Lambda^{\times}(X^*)$ such that

$$\sup_{i\geq 1} |\langle x, Z^*_{ij}(g_i)\rangle| \leq |\langle x, f_j\rangle|, \ \forall\ j \geq 1 \text{ and } x \in X.$$

PROOF : (i) <==> (ii). Since $(\mu(Y),\sigma(\mu(Y),\mu^{\times}(Y^*)))$ is sequentially complete by remark of Proposition 2.1, Chapter 3 and Proposition 3.5, Chapter 2, the equivalence of (i) and (ii) follows from Theorem 2.3 and Proposition 2.4.

(ii) ==> (iii). It is a direct consequence of Theorem 5.1.

(iii) ==> (i). In order to prove (i), let us consider a point $\bar{x}$ in $\Lambda(X)$ and $\bar{g}$ in $\mu^{\times}(Y^*)$ Then from (iii), we can find $\bar{f} \in \Lambda^{\times}(X^*)$ such that

$$|\langle x_j, Z^*_{ij}(g_i)\rangle| \leq |\langle x_j, f_j\rangle|, \ \forall\ i,j \geq 1.$$

$$(*) \quad \Longrightarrow \quad \sum_{j\geq 1} |\langle x_j, Z^*_{ij}(g_i)\rangle| = \sum_{j\geq 1} |\langle Z_{ij}(x_j), g_i\rangle| \leq \sum_{j\geq 1} |\langle x_j, f_j\rangle| < \infty ,$$

for each $i \geq 1$. In particular, choosing $\bar{g} = \delta_i^g$ for $g \in Y^*$ and arbitrarily fixing $i \geq 1$, we derive the weak convergence of $\sum_{j\geq 1} Z_{ij}(x_j)$ to some point y_i in Y from (*) and $\sigma(Y,Y^*)$-sequential completeness of Y.

Now, to establish the proof of (i), it remains to show that $\bar{y} = \{y_i\}$ is a member of $\mu(Y)$. For this, consider $\bar{g} \in \mu^\times(Y^*)$. Then by the nuclearity of $\mu(Y)$ and Proposition 4.6, Chapter 2, we can find $\{\alpha_i\} \in \ell^1$, $\bar{h} = \{h_i\} \in \mu^\times(Y^*)$ with $g_i = \alpha_i h_i$, $i \geq 1$. Applying (iii), for $\bar{h} \in \mu^\times(Y^*)$, there exists $\bar{f}$ in $\Lambda^\times(Y^*)$ such that

$$|\langle x, Z_{ij}^*(h_i)\rangle| \leq |\langle x, f_j\rangle|, \ \forall i,j \geq 1 \text{ and } x \in X.$$

Then

$$\begin{aligned}
\sum_{i\geq 1} |\langle Z(\bar{x})_i, g_i\rangle| &= \sum_{i\geq 1} \Big| \sum_{j\geq 1} \langle Z_{ij}(x_j), g_i\rangle \Big| \\
&\leq \sum_{i\geq 1} \sum_{j\geq 1} |\langle Z_{ij}(x_j), \alpha_i h_i\rangle| \\
&= \sum_{i\geq 1} |\alpha_i| \sum_{j\geq 1} |\langle x_j, Z_{ij}^*(h_i)\rangle| \\
&\leq \sum_{i\geq 1} |\alpha_i| \sum_{j\geq 1} |\langle x_j, f_j\rangle| \\
&< \infty
\end{aligned}$$

$\Longrightarrow Z(\bar{x}) \in \mu^{\times\times}(Y)$. As $\mu(Y)$ is perfect, (i) follows.

Hence we have (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii).

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